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Investigations on the basic theory of STATIC AND DYNAMIC PRESSURE PHENOMENA IN GRAIN under conditions of storage

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FOREWORD

The Transportation and Facilities Research Division of the Agricultural Research Service conducts research to find ways to hold down the costs of physical distribution of products between farms and consumers. It seeks to determine and bring about the adoption of the most efficient facilities, equipment, and methods for moving these products through distributive channels.

During the last 30 years, as grain elevators have become much larger, there have been a number of structural failures. These failures apparently have been caused by excessive pressures on the walls of bins or silos by bulk-stored grain. Many of these failures have resulted in substantial financial losses. Conversely, there is reason to believe, because of inadequate data on the pressures exerted by bulk-stored grain under dynamic conditions, that many elevators constructed in recent years have a built-in "safety factor" that has unnecessarily increased construction costs. Most elevator design engineers still compute the structural strengths required in these facilities on the theories of static pressures proposed by Janssen in 1895 and modified by Köenen in 1896. In 1966, research was initiated under a contract with Dr. Joel D. Isaacson, St. Louis, Mo., to devise a method, through the use of analytic and algebraic models, for computing the pressures exerted by bulk-stored grain under dynamic conditions. This publication contains the contractor's report as it was written. The views expressed are those of the contractor and do not necessarily represent the views of the Division and the Department of Agriculture.

William C. Crow
Director
Transportation and Facilities
Research Division
Agricultural Research Service
U.S. Department of Agriculture

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

INVESTIGATIONS ON THE BASIC THEORY OF STATIC AND DYNAMIC PRESSURE PHENOMENA IN GRAIN UNDER CONDITIONS OF STORAGE

By

Joel D. Isaacson

CHAPTER 1

INTRODUCTION AND SUMMARY

1.1. INTRODUCTION

The determination of static and dynamic pressures exerted by grain on the containing deep bin structure continued to be a hazardous task for the designing engineer. Mathematical and computer investigations on the basic theory underlying grain pressure mechanisms are presented in this report.

1.1.1. The Nature of the Problem

Grain pressures in deep bins were first calculated as for a semi-liquid of the same density as the grain. Thus, for example, the lateral pressure was calculated from the hydrostatic formula,

$$p(y) = \gamma y \tag{1.1}$$

where,

$p(y)$ - the lateral pressure exerted by the grain on the wall at the depth y

γ - the density of the grain

y - the depth of the cross-section calculated.

This method was inadequate because many structures buckled under the vertical load arising from the friction of the grain on the walls - a phenomenon unknown at the time. On the other hand, it was realized later that the calculations based on hydrostatic pressure distribution yielded exaggerated results compared with the actual lateral pressure.

The developments in the field of soil mechanics for calculating earth pressure gave rise to the definition of a new factor, k_o, "the coefficient of earth pressure at rest", which was borrowed for use in the grain pressure problem. Equation (1.1) thus became,

$$p(y) = k_o \gamma y \tag{1.2}$$

where k_o was usually determined according to Rankine as

$$k_o = \frac{1-\sin\theta}{1+\sin\theta} \tag{1.3}$$

where θ is the angle of internal friction of the grain. This improved the estimation of the lateral pressure, but still did not take into account the vertical load on walls. The whole weight of the grain was erroneously assumed to be transmitted to the bottom.

The next development was proposed by Janssen (1895)

and took into account the grain-wall friction.

$$p(y) = \frac{R\gamma}{\mu}\left[1-\exp\left\{-\left(\frac{\mu k_o}{R}\right)y\right\}\right] \qquad (1.4)$$

where,

R - the "hydraulic radius" of the bin cross-section

μ - the coefficient of friction of the grain on the wall

k_o - Rankine coefficient of earth pressure at rest, proposed for this case by Köenen (1896)

Since that time a considerable number of formulas have been proposed by various investigators. Most of them give similar results to that of the Janssen-Köenen method, and consequently the latter has remained in widest use.

Over a long period it appeared that the problem has been solved; however, during the last three decades, when grain elevators became much larger and more numerous, some failed due to excessive pressures. Investigations have shown that during discharge operations so called "dynamic pressure" develop that may be twice as high as the calculated Janssen's pressure. See Figure 1.1.

Whereas the conventional formulas are more or less applicable to static conditions, there do not exist satisfactory methods of estimating grain pressures under

1 - Hydrostatic pressure curve, Eq. (1.1)

2 - Earth pressure curve, Eq. (1.2)

3 - Conventional pressure curve (Janssen's), Eq. (1.4)

4 - Typical empirical dynamic pressure curve

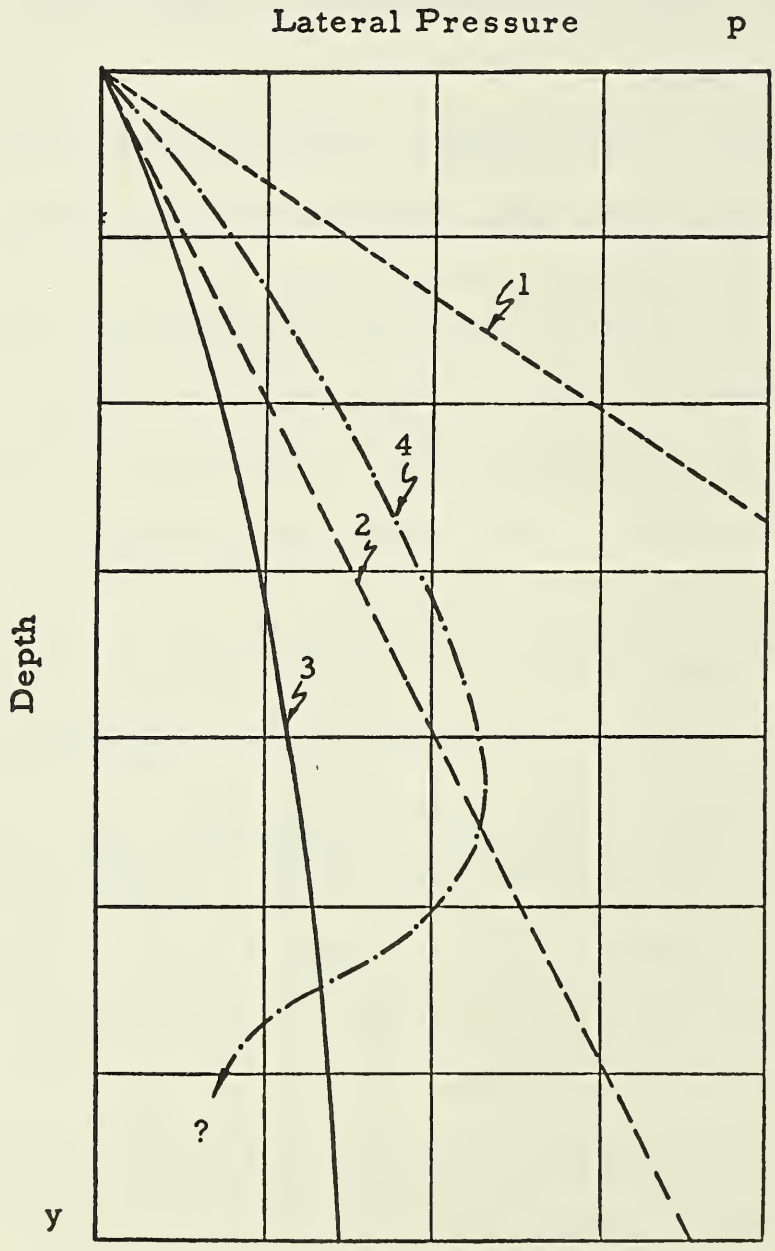


Figure 1.1 Geometrical representation of the nature of the problem.

dynamic conditions. An attempt to rectify this situation is made in this report.

1.2 SUMMARY

The following is a brief summary of topics treated in this report.

Chapter 2 is introductory, giving general definitions of factors associated with grain pressure systems. It also defined deep bin in quantitative terms, and devises a compact classification of granular materials.

In chapter 3, the basic one-dimensional pressure mechanism is formulated, first in analytic and then in algebraic terms. The analytic model is based on the solution of an ordinary linear differential equation of the first order with (possibly) variable coefficients. A practical solution of the analytic model is given, accompanied by a computer program.

The algebraic model is given in a compact form, involving series of determinants and matrices, readily applicable to numerical calculations. The models are very general, allowing both static and dynamic conditions, varied bin geometries, and surcharge/no surcharge conditions. They are compared for various cases. For the case of constant coefficients the analytic and the algebraic models are shown to be mutually identical as well as identical with

Janssen's solution. For variable coefficients (a linearly increasing density-function, a hyperbolically decreasing ratio-function, and a constant wall-friction parameter) the models stay identical. A special choice of the parameters leads to complete identity with the Reimberts' semi-empirical solution. The models are shown to be prototypes of many solutions, from multi-dimensional dynamic to the one-dimensional static, the Reimberts', Janssen's, Rankine's and the hydrostatic solutions.

The characteristic functions are discussed in chapter 4. These are the density, ratio and friction functions, giving the unit weight, the ratio between the lateral to the vertical pressures, and the ratio between the frictional stress and the lateral pressure, as functions of the depth coordinate. Typical characteristic functions associated with static and dynamic conditions are derived and analyzed. Some practical recommendations follow.

Numerical and computer models are given in chapter 5. Three examples of numerical models based on the analytic model are given first. The models are complete with computer programs and printed results. Next are given four examples of computerized graphic representations of various analytic models. Lastly, six examples of digital simulations based on the algebraic model. These models are also complete with computer programs and printed results.

Chapter 6 outlines methods for utilizing the basic one-dimensional models to solve models of higher complexities. The first section gives formal generalizations of the one-dimensional models and characteristic functions. In section 6.2 a computer-oriented method of solving two-dimensional problems by iterative use of one-dimensional models is introduced. Section 6.3 discusses the complexities introduced into the grain pressure system due to feedback and coupling mechanisms. The problem is handled through graph-theoretic techniques. Problems of this category can be represented by directed graphs and solved by controlled iteration of one-dimensional problems.

Chapter 7 introduces an approach based on topological considerations. A brief presentation of the principles leads to intermediate results which show that the maximum dynamic lateral pressure in typical circular cylindrical grain bins should occur at about one-third of the total height H above the bottom and decrease sharply toward a point about 0.12 H above the bottom. The last section describes briefly a computer system developed under this contract to produce grain pressure simulations in the form of 16mm motion-pictures.

Appendix A gives an exhaustive bibliography on grain pressure. The literature survey covers material published since 1883 on a world-wide basis. Among the

countries represented in the survey are (in alphabetical order): Algeria, Argentina, Austria, Canada, England, France, Germany, Hungary, Ireland, Israel, Italy, Mexico, the Netherlands, Poland, South Africa, the Soviet Union, Sweden and the U.S.

Appendix B gives a complete program listing of a package for automatic three-dimensional plots.

CHAPTER 2

DEFINITIONS

2.1 GENERAL DEFINITIONS

(1) A bin is a container that bounds geometric figures belonging to the set of all right cylinders that satisfy either of the following conditions:

a. The directrix of the cylindrical surface is an arbitrary closed convex curve.

b. The directrix is a pair of infinite parallel straight lines; the bin is then called an infinite bin.

The cylindrical surface is called the wall(s) or boundary of the bin.
The lower base is called the (flat) bottom.
The upper base is called the top, and physically may or may not exist.

(2) The hydraulic radius of a bin is in the respective cases above:

a. The ratio of the area bounded by the closed directrix to the length of the directrix.

b. The ratio of the area bounded by the rectangular cross-section of a finite segment of an infinite bin, to the total length of the walls included in the segment.

(3) A bin is shallow or deep according to whether the ratio of its height to the hydraulic radius is less than or equal to or larger than some arbitrary positive real number, respectively.

(4) A bin is called a grain bin if its interior is partially or completely occupied by grain. (See def. 6.)

(5) Granular material is a mass consisting of a collection of solid particles and has the following properties:

a. A fixed natural angle of repose. (See def. 7.)

b. If bounded by a deep bin, it maintains gravity flow when some region of the bottom is removed.

c. If bounded by a deep bin, it transforms to a geometric figure of constant slopes when the walls are gradually removed.

(6) Grain is a collection of plant seeds that satisfies the properties of granular materials. A s-ngle member of the collection is called a kernel.

(7) The natural angle of repose of a granular material is the angle of the steepest slope of the right circular cone with horizontal base that the granular material may form under gravity.

(8) The lateral pressure at a point on the wall surface is the sum of the horizontal forces exerted by the grain on a square neighborhood of this point, of unit area.

(9) The frictional stress at a point on the wall surface is the sum of the vertical forces exerted by the grain (due to lateral pressure at this point) on a square neighborhood of this point, of unit area.

(10) The wall friction parameter at a point on the wall surface at a given instant, is the ratio between the frictional stress and the lateral pressure at this point, at the given instant.

(11) The vertical pressure at a point on a horizontal cross-section of the grain is the sum of the vertical downward forces exerted by the grain on a square neighborhood of this point, of unit area.

(12) The bottom pressure is the vertical pressure on the bottom.

2.2 DEFINITION OF A DEEP BIN

The wall friction effect under static conditions characterizes the main difference between deep and shallow bins. It is therefore desirable to use the wall friction effect as a measure of the "deepness" of a bin. For this purpose Janssen's equation may be used.

The asymptotic behavior of the pressure is due to the exponential function

$$\xi(y) = \exp\{ -(\frac{\mu k_o}{R})\, y\} \tag{2.1}$$

where $0 \leq (y) \leq 1$ for $0 \leq y \leq \infty$.

By definition, a bin is <u>deep</u> if its total depth H satisfies the inequality $H \geq H_d$, where H_d is a depth such that $\xi(H_d) = 0.05$.

In other words, a bin can be considered to be deep if its depth is sufficiently large such that at least 95 percent of the wall friction effect is developed at some level above the bottom, and any load below this level is essentially carried by the walls alone, due to friction.

To solve for the ratio H_d/R, insert H_d in Eq.(2.1),

$$\xi(H_d) = \exp\{ -(\frac{\mu k_o}{R})\, H_d\} = 0.05$$

and thus

$$\frac{H_d}{R} = \frac{3}{\mu k_o} \quad . \tag{2.2}$$

For example, in terms of the ratio H_d/D for circular cylindrical bins, one obtains

$$\frac{H_d}{D} = \frac{3}{4} \times \frac{1}{\mu k_o} \quad . \tag{2.3}$$

Typical values of k_o and μ for grain are respectively 0:333 and 0.450, and therefore

$$\frac{H_d}{D} = \frac{3}{4} \times \frac{1}{0.450 \times 0.333} = 5,$$

so that for practical considerations a circular cylindrical grain bin is <u>deep</u> if

$$\frac{H}{D} \geq 5. \tag{2.4}$$

2.3 CLASSIFICATION OF MATERIALS

Granular materials are sometimes referred to as semi-fluids or bulk-solids, probably to suggest their intermediate properties. It seems that granular materials subject to gravity pressures indeed possess properties whose ranges lie between those of fluids and monolithic solids. Table 2.1 presents a classification of materials with respect to gravity pressures in storage containers. From this classification there arises a compact formulative way to designate materials by their gravity-pressure properties. For example, a material satisfying the hydrostatic equation for lateral pressure is designated by I-A1-B1-C1-D1-E1, i.e. water. This kind of designation is called a <u>material-type</u>. More than one material is associated with a material-type. In general, higher order characters in a material-type indicate more complex materials.

Table 2.2 presents some examples of material-types.

TABLE 2.1 Classification of materials with respect to gravity pressures in storage containers

MATERIAL	RANGE OF VALUES									
	A		B		C		D		E	
	k		λ		θ		c		γ	
	1	2	1	2	1	2	1	2	1	2
	const.	var.	const.	var.	const.	var.	const.	var.	const.	var.
I fluids	$k=1$	---	$\lambda=o$	---	$\theta=o$	---	$c=o$	---	$\gamma=\gamma_o$	---
II non-cohesive granular materials	$o\leq k\leq 1$	$o\leq k\leq 1$	$o\leq\lambda\leq 1$	$o\leq\lambda\leq 1$	$o<\theta<\frac{\pi}{2}$	$o<\theta<\frac{\pi}{2}$	$c=o$	---	$\gamma=\gamma_o$	$\gamma_o\leq\gamma\leq\gamma_{max}$
III cohesive granular materials	$o\leq k\leq 1$	$o\leq k\leq 1$	$o\leq\lambda\leq 1$	$o\leq\lambda\leq 1$	$o<\theta<\frac{\pi}{2}$	$o<\theta<\frac{\pi}{2}$	$o<c$	$o<c$	$\gamma=\gamma_o$	$\gamma_o\leq\gamma\leq\gamma_{max}$
IV monolithic solids	$k=o$	---	$\lambda=o$	---	$\frac{\pi}{2}\leq\theta$	---	---	---	$\gamma=\gamma_o$	---

Where:
- k - ratio between lateral and vertical pressures at any level
- λ - ratio between frictional stress and lateral pressure at any level
- θ - natural angle of repose
- c - cohesion
- γ - unit weight

TABLE 2.2 Examples of material-types of materials stored in bins

	Material-type	Material
a	I-A1-B1-C1-D1-E1	water, oil
b	II-A1-B1-C1-D1-E1	dry sand, aggregate, Janssen's grain
c	II-A2-B2-C2-D1-E2	grain
d	III-A2-B2-C2-D1-E1	Portland cement, flour
e	III-A2-B2-C2-D2-E2	silage
f	IV-A1-B1-C1-D0-E1	conglomerated grain, hardened concrete 1/

1/ Materials may change material-types due to chemical, organic, biological and environmental factors. An extreme example is concrete. When poured into the forms it is a mixture of components (water, sand, aggregate and cement) belonging to material-types (a), (b), (b), (d), respectively. Nevertheless, the first component is qualitatively dominant and poured concrete is closest to material-type (a). However, after a relatively short period the concrete hardens and gradually becomes material-type (f).

Similar changes, although less critical, may occur in grain, flour or silage, under certain conditions.

The concerns of this study are grain-types: II-A2-B2-C2-D1-E2 and simpler types.

CHAPTER 3

BASIC PRESSURE MECHANISM

3.1 ANALYTIC MODEL

3.1.1 Introduction

For clarity, the one-dimensional problem is dealt with first. In the one-dimensional problem all variables are considered functions of the depth coordinate - y - alone. For simplicity, the infinite bin case is considered. Similar treatment can be applied to any other bin with a symmetrical cross-section, such as a circle, a square, or a regular polygon.

3.1.2 Derivation of the Governing Differential Equation

Consider a horizontal differential slice of height dy at depth y in a unit segment of an infinite bin, having width 2ρ. The vertical pressure acting on the top surface of the prism is q, the lateral pressure exerted by the prism of grain on the walls is p, and the resulting frictional stress is s. Let dq denote the increase in vertical pressure along the interval $(y, y + dy)$. Then the increase in the resultant of the vertical forces acting on the prism in a downward direction, along this interval, is

$$A\,dq = \gamma A\,dy - s\,L\,dy \qquad (3.1)$$

where

A - cross-sectional area

L - that portion of the cross-sectional perimeter which is bounded by the walls

γ - unit weight of grain

Rearrangement of Eq. (3.1) yields

$$\frac{dq}{dy} + s\,\frac{L}{A} = \gamma. \qquad (3.2)$$

Define

$$k = \frac{p}{q} \qquad (3.3)$$

$$\lambda = \frac{s}{p} \qquad (3.4)$$

$$R = \frac{A}{L} \qquad (3.5)$$

and thereby

$$s = \lambda\, k\, q. \qquad (3.6)$$

Insert Eq. (3.6) into Eq. (3.2) to obtain

$$\frac{dq}{dy} + \frac{1}{R}\,\lambda\, k\, q = \gamma \qquad (3.7)$$

and assume that R, k, λ, γ are constants or functions of y alone. Denote

$$\beta = \frac{\lambda k}{R}. \qquad (3.8)$$

Then, the most general equation is of the form

$$\frac{dq}{dy} + \beta(y)q = \gamma(y). \qquad (3.9)$$

Eq. (3.9) is an ordinary linear differential equation of the first order, where y and q are the independent and dependent variables respectively, and β and γ are functions of y alone, or constants.

3.1.3 Initial Conditions and Solutions

To find the general solution of Eq. (3.9) consider first the homogeneous linear equation

$$\frac{dq}{dy} + \beta q = 0.$$

Its variables are separable, thus

$$\frac{dq}{q} + \beta dy = 0$$

and the solution is

$$q = c \exp(-\int \beta dy)$$

where c is a constant.

Now substitute in the non-homogeneous equation, the expression

$$q = w \exp(-\int \beta dy)$$

in which w, a function y, replaced the constant c. The equation becomes

$$\frac{dw}{dy} \exp(-\int \beta dy) = \gamma,$$

whence

$$w = C + \int \gamma \exp(\int \beta dy)\, dy.$$

The general solution is therefore

$$q = C \exp(-\int \beta dy) + \exp(-\int \beta dy) \int \gamma \exp(\int \beta dy)\, dy \qquad (3.10)$$

where C is an arbitrary constant. In order to determine C uniquely, an initial condition must be specified. In this case (a one-dimensional problem), it is necessary and sufficient to specify at the initial point y = 0, the following initial conditions:

$$q(0) = \begin{cases} 0 & \text{without surcharge} \\ 1/2\gamma_o h & \text{with triangular surcharge of height } h; \ \gamma_o \text{ is constant.} \end{cases} \qquad (3.11)$$

(1) <u>Solution without surcharge</u>. Imposing the first initial condition on Eq. (3.10) one obtains

$$C = -\int \gamma \exp(\int \beta\, dy)\, dy \Big|_{y=o} \qquad (3.12)$$

and by Eqs. (3.3), (3.10), and (3.12), a complete solution for the lateral pressure is of the form

$$p = k \exp(-\int \beta\, dy)\left[\int \gamma \exp(\int \beta\, dy)\, dy - \int \gamma \exp(\int \beta\, dy)\, dy \Big|_{y=o}\right] \qquad (3.13)$$

(2) <u>Solution with surcharge</u>. Imposing the second

initial condition on Eq. (3.10) one obtains

$$C = \{1/2\gamma_o h \exp(\int \beta \, dy) - \int \gamma \exp(\int \beta \, dy) \, dy\}\Big|_{y=o} \qquad (3.14)$$

and a complete solution is of the form

$$p = k \exp(-\int \beta \, dy)\left[\int \gamma \exp(\beta \int dy) \, dy + \{1/2\gamma_o h \exp(\int \beta \, dy) - \int \gamma \exp(\int \beta \, dy) \, dy\}\Big|_{y=o}\right]. \qquad (3.15)$$

Notes. By the broad definition of L and A the solution is in fact applicable not only to the infinite bin case, but also to any symmetrical bin. Since case (1) is less complicated, unless otherwise indicated Eq. (3.13) will be hereinafter referred to as the analytic model.

Eq. (3.13) gives a complete analytic solution to the one-dimensional problem, provided β and γ are known and can be expressed in rather simple forms. It will be shown later (see Chapter 4) that plausible assumptions can be made regarding the forms of these functions. However, in general, it is best to solve Eq. (3.13) numerically using a procedure that admits arbitrary forms of β and γ. A procedure such as this is presented in the next section.

3.2 NUMERICAL SOLUTION (RUNGE-KUTTA) FOR THE ANALYTIC MODEL

3.2.1. Introduction

Initial value problems can be stated as follows:

Given

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) & (3.16) \\[2ex] \text{I.C.: } y(x_o)=y_o, & (3.17) \end{cases}$$

find a solution $y = g(x)$, such that $y_o=g(x_o)$. (Here, x and y are the independent and dependent variables, respectively).

To express the grain pressure problem in these terms it is necessary to rewrite Eqs. (3.9) and (3.11) as follows,

$$\begin{cases} \dfrac{dq}{dy} = \gamma(y) - \beta(y)\, q & (3.18) \\[2ex] q(0) = \text{constant} & (3.19) \end{cases}$$

The R.H.S. of Eq. (3.18) can be any arbitrary function of y and q, thus allowing a great flexibility in choosing β and γ. The constant in the initial condition depends on the surcharge. It is usually zero.

With the advent of modern high-speed digital computers, numerical methods for the solution of initial value problems in ordinary differential equations have found widespread application. A variety of methods and special-purpose simulation languages are available. Benyon's article (1) on numerical methods for digital simulation reviews many of these methods. 1/ Further information can be found in references (2) through (42). After considerable investigation it was found that for the purpose of solving the analytic model, a Runge-Kutta method is well-suited. A brief description of the method follows.

3.2.2 A Fourth-Order Runge-Kutta Method

Runge-Kutta methods involve essentially replacing a truncated Taylor's series expansion of the solution, given in terms of derivatives, by an approximation in terms of $f(x,y)$ only. The derivation of the method can be found in many books on numerical analysis, for example (24).

Using a fourth-order Runge-Kutta integration procedure the ordinary differential equation, $dy/dx=f(x,y)$ with initial condition $y(x_o) = y_o$ is solved numerically. This is a single-step method in which the value of y at $x = x_n$ is used to compute $y_{n+1} = y(x_{n+1})$ and earlier values y_{n-1}, y_{n-2}, etc. are not used.

1/ Underscored numbers in parentheses refer to References, p. 150.

The relevant formulas are:

$$y_{n+1} = y_n + 1/6\ (k_o + 2k_1 + 2k_2 + k_3) \qquad (3.20)$$

Where, for step size h:

$$\begin{cases} k_o = hf(x_n, y_n) \\ k_1 = hf(x_n + h/2,\ y_n + k_o/2) \\ k_2 = hf(x_n + h/2,\ y_n + k_1/2) \\ k_3 = hf(x_n + h,\ y_n + k_2) \end{cases} \qquad (3.21)$$

There formulas were programmed and implemented on the computer. A subroutine RUNKUT follows.

```
C     SUBROUTINE RUNKUT
C
C     PURPOSE
C        INTEGRATES THE FIRST ORDER DIFFERENTIAL EQUATION OF THE
C        MODEL, DY/DX=FUN(X,Y) AND PRODUCES A TABLE OF INTEGRATED VALUES
C
C
C     USAGE
C        CALL RUNKUT(FUN,H,XI,YI,K,N,VEC)
C
C     DESCRIPTION OF PARAMETERS
C
C        FUN-SUPPLIED FUNCTION SUBPROGRAM WITH ARGUMENTS X,Y
C            (DEPTH AND LATERAL PRESSURE) WHICH GIVES DY/DX
C        H  -STEP SIZE ALONG DEPTH COORDINATE
C        XI -INITIAL VALUE OF X (ZERO USUALLY)
C        YI -INITIAL VALUE OF Y WHERE YI=Y(XI) (USUALLY ZERO,
C            OR DEPENDS ON THE SURCHARGE)
C        K  -THE INTERVAL AT WHICH THE COMPUTED VALUES ARE TO BE STORED
C        N  -THE NUMBER OF VALUES TO BE STORED
C        VEC-THE RESULTANT VECTOR OF LENGTH N IN WHICH COMPUTED
C            VALUES OF Y ARE TO BE STORED
C
C
C
C
      SUBROUTINE RUNKUT(FUN,H,XI,YI,K,N,VEC)
      DIMENSION VEC(1)
      H2=H/2.
      Y=YI
      X=XI
      DO 2 I=1,N
      DO 1 J=1,K
      T1=H*FUN(X,Y)
      T2=H*FUN(X+H2,Y+T1/2.)
      T3=H*FUN(X+H2,Y+T2/2.)
      T4=H*FUN(X+H,Y+T3)
    1 X=X+H
    2 VEC(1)=Y
      Y= Y+(T1+2.*T2+2.*T3+T4)/6.
      RETURN
      END
```

3.3 ALGEBRAIC MODEL

In addition to the analytic model, it seems desirable to supply an alternative algebraic solution that is free from any analytical difficulties that may normally be associated with a complicated expression such as Eq. (3.13). Another reason for seeking an algebraic solution is to make practical calculations more readily applicable to programming and processing by digital computers.

3.2.1 Construction of the Basic Model

Consider, again, a unit segment of the infinite bin having width and height 2ρ and H, respectively. For reasons of symmetry, treat one-half only. Divide H into u slices each of height Δy. Starting from the top, associate with each slice an ordinal number i, ($1 \leq i \leq u$, i is an integer), and local values for various parameters, such as: unit weight γ_i, wall-friction parameter λ_i, and ratio parameter K_i.

NOMENCLATURE:

γ_i - average unit weight of the i-th slice

A - cross-sectional area, $A=1.\ \rho= \rho$

G_i - weight of the i-th slice, $G_i = \gamma_i A \Delta y$

Q_i - resultant of the downward vertical forces acting on the i-th slice

F_i - resultant of the lateral forces acting on the i-th slice

S_i - resultant of the vertical frictional forces exerted by the i-th slice on the wall due to F_i

K_i - ratio between F_i and Q_i, $K_i = F_i/Q_i$

λ_i - wall friction parameter at the i-th slice (ratio between S_i and F_i, $\lambda_i = S_i/F_i$)

q_i - average vertical pressure on a horizontal cross-section passing through the center of gravity of the i-th slice

p_i - average lateral pressure exerted by the i-th slice on the wall

s_i - average frictional stress at the wall, exerted by the i-th slice

k_i - ratio between p_i and q_i, $k_i = p_i/q_i$

T_i - accumulated load per unit length carried by the wall at the i-th slice

ρ - hydraulic radius

Isolate slice #1 and examine the forces acting upon it:

(i) weight of slice - G_1 (downwards)

(ii) resultant of lateral forces - $F_1 = K_1 G_1$ (inwards)

(iii) resultant of frictional forces - $S_1 = \lambda_1 K_1 G_1$ (upwards at the wall)

The residual resultant of downward forces is therefore

$$R_1 = G_1(1-\lambda_1 K_1).$$

R_1 is that portion of G_1 that is transmitted to slice #2, the remaining portion being carried by the wall. The same process can be applied to each slice as follows:

slice #	G_i	K_i	λ_i	F'_i	S'_i	R_i
1	G_1	K_1	λ_1	K_1G_1	$\lambda_1K_1G_1$	$G_1(1-\lambda_1K_1)$
2	G_2	K_2	λ_2	K_2G_2	$\lambda_2K_2G_2$	$G_2(1-\lambda_2K_2)$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
i	G_i	K_i	λ_i	K_iG_i	$\lambda_iK_iG_i$	$G_i(1-\lambda_iK_i)$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
u	G_u	K_u	λ_u	K_uG_u	$\lambda_uK_uG_u$	$G_u(1-\lambda_uK_u)$

where F'_i and S'_i are those portions of F_i and S_i, respectively, due to G_i alone.

A portion of the weight of each slice is transmitted to the wall by friction. The residual portion is transmitted downwards to the successive slice. After R_i is transmitted to the i+1 slice, it is again divided into two portions: one is transmitted to the wall and the residual

is transmitted to the i+2 slice, and so on. Thus, the complete process is illustrated as follows: 2/

Slice #	Vertical Downward Forces
1. Slice #1 -	$Q_1 = G_1$

$$
\begin{array}{ll}
\text{2. Slice \#2 -} & \\
\quad \text{from slice \#1 -} & Q_2 = \begin{cases} G_2 \\ G_1(1-\lambda_1 K_1) \end{cases} \\
\text{3. Slice \#3 -} & \\
\quad \text{from slice \#2} & Q_3 = \begin{cases} G_3 \\ G_2(1-\lambda_2 K_2) \\ G_1(1-\lambda_1 K_1)(1-\lambda_2 K_2) \end{cases} \\
\quad \cdot & \cdot \\
\quad \cdot & \cdot \\
\quad \cdot & \cdot \\
\text{i. Slice \#i -} & \\
\quad \text{from slice \#i-1 -} & \\
\quad \text{from slice \#i-2 -} & \\
\quad \cdot & \\
\quad \cdot & \\
\quad \cdot & \\
\quad \text{from slice \#1 -} & Q_i = \begin{cases} G_i \\ G_{i-1}(1-\lambda_{i-1} K_{i-1}) \\ G_{i-2}(1-\lambda_{i-2} K_{i-2})(1-\lambda_{i-1} K_{i-1}) \\ \cdot \\ \cdot \\ \cdot \\ G_1(1-\lambda_1 K_1)(1-\lambda_2 K_2)\ldots(1-\lambda_{i-1} K_{i-1}) \end{cases} \\
\quad \cdot & \cdot \\
\quad \cdot & \cdot \\
\quad \cdot & \cdot
\end{array}
$$

2/ The derivation is made for a case without surcharge. Surcharge may be simply added to G_1.

The sum of the vertical forces acting (downwards) on the n-th slice is therefore

$$Q_n=\Delta y\rho[\gamma_n+\gamma_{n-1}(1-\lambda_{n-1}K_{n-1})+\gamma_{n-2}(1-\lambda_{n-2}K_{n-2})(1-\lambda_{n-1}K_{n-1})+\ldots$$
$$\ldots+\gamma_1(1-\lambda_1K_1)(1-\lambda_2K_2)\ldots(1-\lambda_{n-1}K_{n-1})]. \qquad (3.22)$$

The series in Eq. (3.22) characterizes the model sought. It is desirable to find a more compact form for this series so that it can be handled and analyzed conveniently. The series can be represented as a series of determinants as follows:

$$\gamma_n|1|+\gamma_{n-1}\begin{vmatrix}1 & 0\\ 0 & (1-\lambda_{n-1}K_{n-1})\end{vmatrix}+\gamma_{n-2}\begin{vmatrix}1 & 0 & 0\\ 0 & (1-\lambda_{n-1}K_{n-1}) & 0\\ 0 & 0 & (1-\lambda_{n-2}K_{n-2})\end{vmatrix}+\ldots$$

$$\ldots+\gamma_1\begin{vmatrix}1 & & & & & 0\\ & 1-\lambda_{n-1}K_{n-1} & & & & \\ & & 1-\lambda_{n-2}K_{n-2} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & 1-\lambda_1K_1\end{vmatrix}.$$

The determinants above can be recognized as being associated with the sequence of n submatrices produced from the n-by-n diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1-\lambda_{n-1}K_{n-1} & 0 & & 0 \\ 0 & 0 & 1-\lambda_{n-2}K_{n-2} & & \\ & & & \ddots & \\ & 0 & & & 1-\lambda_1K_1 \end{bmatrix}$$

by successively taking the 1-by-1 upper left submatrix, 2-by-2 upper left submatrix, and so on. Let the expression "det Sub_i A" denote "the determinant of the i-th submatrix of an n-by-n arbitrary diagonal matrix A, produced by deleting from A all rows and columns whose ordinal number is larger than i." Then the series of determinants can be written in the form of a compact sum as follows:

$$\sum_{i=1}^{n} \gamma_{n+1-i} \det \mathrm{Sub}_i [I-\lambda K] \qquad (3.23)$$

where I is an n-by-n identity matrix, and λK is a diagonal matrix of the form

$$\lambda K=\text{Diagonal } [0,\lambda_{n-1}K_{n-1},\lambda_{n-2}K_{n-2},\ldots,\lambda_1K_1]. \qquad (3.24)$$

Thus Eq. (3.22) becomes

$$Q_n=\Delta y\rho \sum_{i=1}^{n} \gamma_{n+1-i} \det \text{Sub}_i [I - \lambda K] \qquad (3.25)$$

and other quantities are readily obtainable in terms of Eq. (3.25) as follows:

(i) Average vertical pressure at the n-th slice

$$q_n = \frac{Q_n}{\rho}$$

$$= \Delta y \sum_{i=1}^{n} \gamma_{n+1-i} \det \text{Sub}_i [I - \lambda K]. \qquad (3.26)$$

(ii) Average lateral pressure exerted on the wall at the n-th slice

$$p_n = k_n q_n; \qquad k_n = \frac{K_n \rho}{\Delta y} \qquad (3.27)$$

$$= K_n \rho \sum_{i=1}^{n} \gamma_{n+1-i} \det \text{Sub}_i [I - \lambda K]. \qquad (3.28)$$

(iii) Average frictional stress at the wall at the n-th slice

$$s_n = \lambda_n p_n$$

$$= \lambda_n K_n \rho \sum_{i=1}^{n} \gamma_{n+1-i} \det \text{Sub}_i [I - \lambda K]. \qquad (3.29)$$

(iv) Accumulated load per unit length carried by the wall at the n-th slice

$$T_n = \sum_{j=1}^{n} s_j$$

$$= \sum_{j=1}^{n} \lambda_j K_j \sum_{i=1}^{j} \gamma_{j+1-i} \det \mathrm{Sub}_i[I - \lambda K]. \quad (3.30)$$

3.4 COMPARISON BETWEEN THE MODELS AND RELATIONSHIPS TO OTHERS

Clearly the two models were mathematically derived differently, though both represent the same physical situation. It is of interest to carry out a preliminary investigation, comparing between the models, as well as with known models such as Janssen's and the Reinberts'.

3.4.1 Constant Coefficients

Start out with the simplest case where all parameters are constant. Let $\lambda = \mu$, k, γ, and ρ be constants. Also, let all the non-zero elements of the diagonal matrix λK be the same, such that $(\lambda K)_{ii} = \mu K$, where $K = \frac{k}{\rho} \Delta y$ is constant.

All the derivations are with respect to lateral pressure only.

(1) Analytic solution. The lateral pressure p is given by Eq. (3.13). Substitute the constant coefficients into Eq. (3.13) to obtain

$$p = k \exp(-b\int dy)\left[\gamma\int \exp(b\int dy)\,dy - \gamma\int \exp(b\int dy)\,dy\Big|_{y=o}\right]$$

where $b = \frac{\mu k}{\rho}$ is a constant. Then

$$\begin{aligned} p &= k\gamma \exp(-by)\,[\tfrac{1}{b}\exp(by) - \tfrac{1}{b}] \\ &= \frac{k\gamma}{b}\,[1 - \exp(-by)] \\ &= \frac{\rho\gamma}{\mu}\,[1 - \exp\{-(\tfrac{\mu k}{\rho})y\}]. \end{aligned} \tag{3.31}$$

It is easy to observe that Eq. (3.31) turns out to be identical with the well-known Janssen's solution.

(2) Algebraic solution. From Eq. (3.28), the lateral pressure p_n can be written

$$p_n = K_n \rho \sum_{i=1}^{n} \gamma_{n+1-i} \det \mathrm{Sub}_i [I - \lambda K]. \tag{3.32}$$

Substitute the constant parameters in Eq. (3.32) to obtain

$$p_n = K\rho\gamma[1 + (1-\mu K) + (1-\mu K)^2 + \ldots + (1-\mu K)^{n-1}].$$

The series included in the square brackets is a geometric series with a ratio factor smaller than 1. Summing the geometric series

$$p_n = K\rho\gamma\left[\frac{1-(1-\mu K)^n}{1-(1-\mu K)}\right]$$

$$= \frac{\rho\gamma}{\mu}[1-(1-\mu K)^n]. \tag{3.33}$$

Using Eq. (3.27)

$$K = \frac{k}{\rho} \cdot \frac{y}{n}, \tag{3.34}$$

and (3.33) finally becomes

$$p_n = \frac{\rho\gamma}{\mu}[1-\{1-(\frac{\mu k}{\rho})y\frac{1}{n}\}^n]. \tag{3.35}$$

(3) Comparison with Janssen's solution. Compare between Eq. (3.31) and Eq. (3.35) to find out that except for the expressions $\exp\{-(\frac{\mu k}{\rho})y\}$ and $\{1-(\frac{\mu k}{\rho})y\frac{1}{n}\}$, the equations

are identical. Now, using a limiting process where $n \to \infty$ $(\Delta y \to o)$, it is evident that

$$\lim_{\substack{n\to\infty \\ (\Delta y \to o)}} \{1 + \frac{1}{n}(-\frac{\mu k}{\rho})y\}^n = \exp\{-(\frac{\mu k}{\rho})y\}.$$

It is thus shown that Eq. (3.31) and Eq. (3.35) become practically identical for sufficiently small Δy. Of course, it was shown in the previous section that Eq. (3.31) is identical with Janssen's solution. Thus it was shown that for the special case of constant parameters both the analytic and algebraic models become identical and simply reduce to the well-known Janssen's solution. This result is very important as it defines the relationship of the new theory to old ones.

3.4.2 Variable Coefficients

An important feature of the new theory is its great flexibility in admitting variable parameters of nearly arbitrary form. A dual test of the models for a special case of variable coefficients follows.

(1) Analytic solution. Explicit integration of the analytic solution Eq. (3.13) is very sensitive to the nature of the functions $\gamma(y)$ and $\beta(y)$, especially the latter. Unfortunately, $\beta(y)$ is bound to be a very complicated function and therefore an analytic solution is

seldom practical. However, considerations explained later (see section 4.3.1) suggest for the qualitative behavior of $\beta(y)$, under certain conditions, the nature of a decreasing hyperbolic function. The simplest form to satisfy this requirement is a rational function of the type

$$\beta(y) = \frac{\eta B}{(1+Cy)} \tag{3.36}$$

where $\eta=\frac{\lambda}{\rho}$, λ and ρ are constants. The ratio -function portion of $\beta(y)$ is then

$$k(y) = \frac{B}{(1+Cy)} \tag{3.37}$$

where B and C are some constants.

The density-function is an increasing function, and for simplicity it is chosen here to be a linear function, for example

$$\gamma(y) = \gamma_o + \delta y \tag{3.38}$$

where γ_o and δ are constants. Insert Eqs. (3.36),(3.38) into Eq. (3.13) and integrate in the three following steps,

(i) Find $\int \beta(y)\, dy = \int \frac{\eta B}{(1+Cy)}\, dy$

$$= \frac{\eta B}{C} \ln(1+Cy) \quad \underline{3}/$$

3/ Constants of integration may be deleted throughout this integration.

(ii) Find $g(y) = \int \gamma(y)\ \exp(\int \beta(y)dy)dy = \int (\gamma_o + \delta y) \exp[N\ \ln(1+Cy)]dy$ where $N = \frac{\eta B}{C}$ is a constant. Expanding and integrating by parts, one finally obtains

$$g(y) = \frac{(1+Cy)^{N+1}}{C(N+1)} [(\gamma_o + \delta y) - \frac{\delta(1+Cy)}{C(N+2)}].$$

(iii) Find $g(o) = \frac{1}{C(N+1)}[\gamma_o - \frac{\delta}{C(N+2)}]$.

Using the results (i), (ii) and (iii) on Eq. (3.13), the solution is

$$p(y) = \frac{B\gamma_o}{C(N+1)}[1 - \frac{1}{(1+Cy)^{N+1}} + \frac{\delta}{\gamma_o}(\frac{N+1}{N+2})y + \frac{\delta}{\gamma_o} \cdot \frac{\{1-(1+Cy)^{N+1}\}}{C(N+2)(1+Cy)^{N+1}}]. \tag{3.39}$$

(2) Algebraic solution. Use the same β and γ functions. In terms of k_n, Eq. (3.28) becomes

$$p_n = k_n\ \Delta y \sum_{i=1}^{n} \gamma_{n+1-i} \det \mathrm{Sub}_i\ [I - \eta k \Delta y] \tag{3.40}$$

where

$$\gamma_{n+1-i} = \gamma_o + \delta[n' + 1 - i]\Delta y; \quad (n'=n - \tfrac{1}{2})$$

$\eta = \frac{\lambda}{\rho}$ is a constant

I is an n-by-n identity matrix

$k_n = \frac{B}{1+Cn'\Delta y}$, and k is the diagonal matrix

$$
k = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & \frac{B}{1+C(n'-1)\Delta y} & & & 0 \\
0 & & \frac{B}{1+C(n'-2)\Delta y} & & \cdot \\
\cdot & & & & \cdot \\
\cdot & & \ddots & & \cdot \\
\cdot & & & & \cdot \\
0 & 0 & \cdots & & \frac{B}{1+C1/2\Delta y}
\end{bmatrix}
$$

(3) Comparison. It is best to compare between Eqs. (3.39) and (3.40) by examining their graphs. A numerical example was worked out, and the results are given in Fig.

The algebraic solution was calculated first for large increments of $\Delta y = 10'$. Twelve discrete points of the solution were obtained and joined by straight lines. When Δy was taken smaller, more points were produced, and the algebraic curve tended to approach the analytic curve. For sufficiently small Δy the curves practically coincided. Thus, it is shown that the two models become identical, in the limit, for variable coefficients as well.

3.4.3 Relationship of the Analytic Model to the Reimberts' Semi-Empirical Solution

Further investigation in this connection led to the

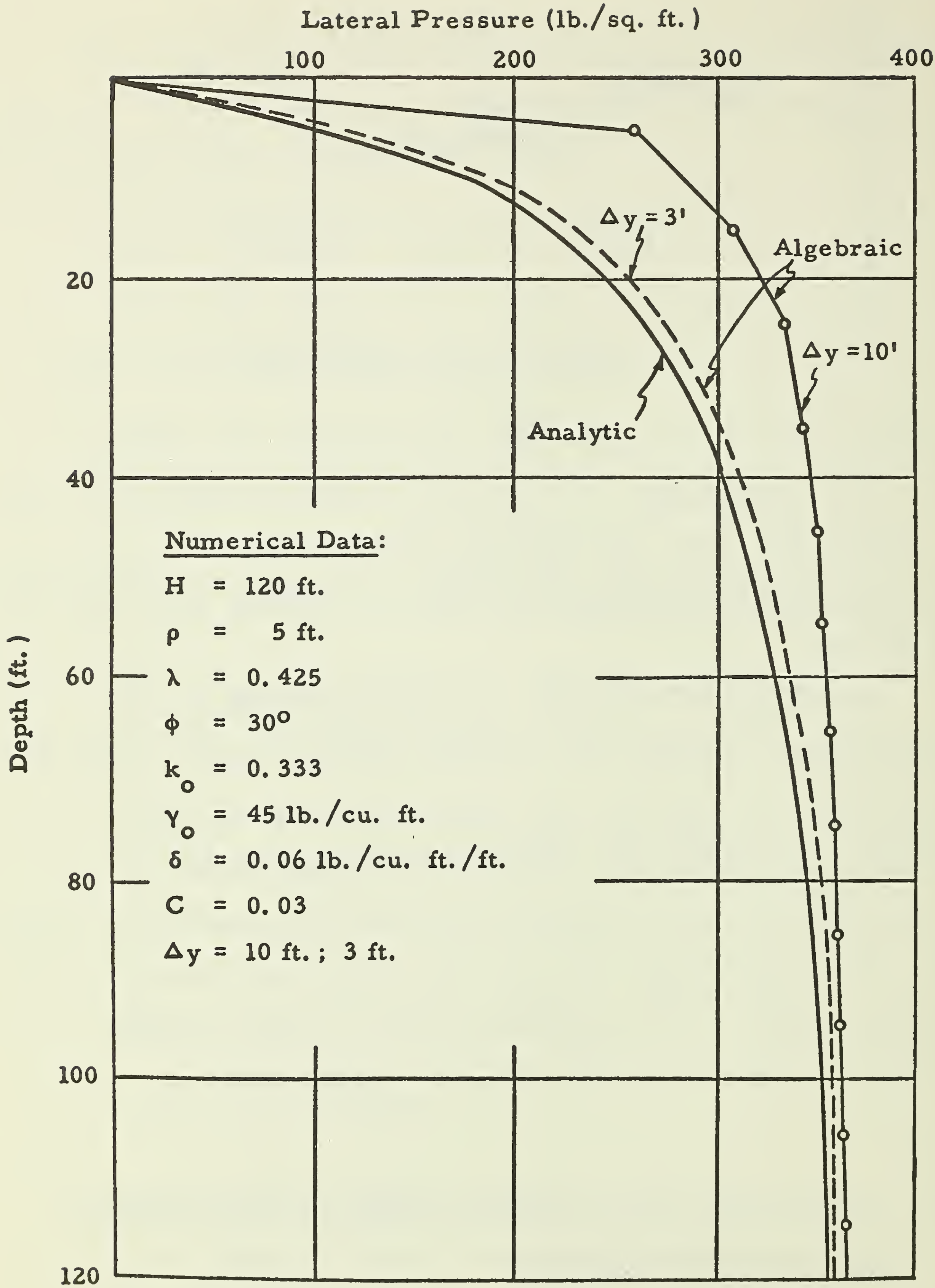


Figure 3.1 Comparison between the analytic and algebraic solutions with variable coefficients.

following important result. Starting with Eq. (3.39), it can be reduced to the Reinberts' semi-empirical solution by an appropriate choice of coefficients and constants.

First, let $\gamma(y) = \gamma_o$ be a constant (i.e. $\delta = 0$), thereby reducing Eq. (3.39) to

$$p(y) = \frac{B\gamma_o}{C(N+1)} \left[1 - (1+Cy)^{-(N+1)}\right]. \tag{3.41}$$

Let $B = 2\,k_o$, where k_o is the Rankine-Köenen-Caquot coefficient, i.e.

$$k_o = \frac{1-\sin\phi}{1+\sin\phi} \tag{3.42}$$

where ϕ is the angle of internal friction of the grain. Let $\lambda = \frac{1}{2}\,\mu$, where μ is the coefficient of grain-wall friction, and let $C = \frac{\mu k_o}{\rho}$. Then

$$N = \frac{\lambda}{\rho} \cdot \frac{B}{C} = \frac{\frac{1}{2}\mu}{\rho} \cdot \frac{2k_o}{\left(\frac{\mu k_o}{\rho}\right)} = 1.$$

Insert the values of N, B, and C into Eq. (3.41) which thus reduces to

$$p(y) = \frac{\rho\gamma_o}{\mu} \left[1-\{1+(\frac{\mu k_o}{\rho})y\}^{-2}\right]. \tag{3.43}$$

It is now easily observed that, except for a difference in notation, Eq. (3.43) is precisely identical to the Reimberts' solution reported in their book (43) (p. 35, Eq. (11)) in the form

$$p_z = p_{max}[1-(\frac{z}{A} + 1)^{-2}]$$

where

z - depth below the edge of the vertical wall

p_z - lateral pressure exerted on the wall by the grain at depth z

p_{max} - $\frac{\delta D}{4tg\phi'}$, where

δ - density of grain

D - interior diameter of cylindrical bin

ϕ' - angle of friction of grain on wall

$A = \frac{D}{4tg\phi' tg^2(\frac{\pi}{4} - \frac{\phi}{2})} - \frac{h}{3}$ is a constant

h - height of a conical surcharge (zero in this comparison)

ϕ - angle of internal friction of grain

$$tg^2(\frac{\pi}{4} - \frac{\phi}{2}) = \frac{1-\sin\phi}{1+\sin\phi} = k_o.$$

Discussion. The Reimberts' solution was derived on the basis of both theoretical considerations and empirical data gathered over years of experimental investigations of dynamic effects in grain bins. The derivation of their method is different from the derivation of the theories of this study, which is strictly theoretical. However, it is most interesting to note that our analysis supplies a fundamental answer to the understanding of the Reimberts' method and results, not known before.

Assuming that the Reimberts' solution is in good agreement with their experimental findings, the result of Eq. (3.43) determines the exact behavior of the ratio-function and wall-friction parameter during their experiments. The essence of the Reimberts' method in relation to ours can be summarized as follows:

(i) the density-function γ_o is a constant

(ii) the wall-friction parameter $\frac{1}{2}\mu$ is a constant

(iii) the ratio-function is a decreasing hyperbolic rational function of the form

$$k(y) = \frac{2k_o}{1+(\frac{\mu k_o}{R})y}$$

(iv) the lateral pressure is calculated according to our general analytic solution in one-dimension, Eq. (3.13), using the above factors as coefficients.

These conclusions are very significant. First, they establish explicit mathematical expressions for a dynamic ratio-function $k(y)$ and a dynamic wall parameter λ. It is confirmed mathematically for the first time that the dynamic ratio-function is a hyperbolically decreasing function. Also, that under dynamic conditions the friction-parameter is one-half the static friction coefficient. It should be noted that without the generalized solution it would have been next to impossible to gain insight into these complex interrelationships under dynamic conditions. Furthermore, at this point, it was established that both the Janssen's and the Reimberts' solutions -- thought until now to be independent -- are not independent, but are simply special cases of the same prototypical model, Eq. (3.13). Many more types of solutions, static and dynamic, can be derived from our model as shown in Table 3.1. Through the use of the digital computer, these models can be realized and then exploited for both research and design purposes.

TABLE 3.1. Analytic solution as a prototype of other solutions.

Material-Type	Characteristic Functions			Analytic Model
	density-function	friction-function	ratio-function	Reduced to:
1. I-A1-B1-C1-D1-E1	$\gamma=\gamma_o$	$\lambda=0$	$k=1$	hydrostatic solution
2. II-A1-B1-C1-D1-E1	$\gamma=\gamma_o$	$\lambda=0$	$k=k_o$	earth pressure solution
3. II-A1-B1-C1-D1-E1	$\gamma=\gamma_o$	$\lambda=\mu$	$k=k_o$	Janssen's solution
4. II-A2-B1-C1-D1-E1	$\gamma=\gamma_o$	$\lambda=\frac{1}{2}\mu$	$k(y)=\dfrac{2k_o}{1+(\frac{\mu k_o}{R})y}$	Reimberts' solution
5. II-A2-B2-C2-D1-E2	$\gamma=\gamma(y)$	$\lambda=\lambda(y,t)$	$k=k(y,t)$	one-dim. dynamic
6. II-A2-B2-C2-D1-E2	$\gamma=\gamma(x,y)$	$\lambda=\lambda(y,t)$	$k=k(x,y,t)$	two-dim. dynamic

CHAPTER 4

THE CHARACTERISTIC FUNCTIONS

4.1 INTRODUCTION

The characteristic functions are:

(1) The density-function $\gamma(y)$

(2) The ratio-function $k(y)$

(3) The friction-function $\lambda(y)$.

They determine the β and γ functions and as such are the building blocks of the basic model. They have to be explicitly specified before a model is fully characterized and implemented. As mentioned before, our models are very general and will admit arbitrary characteristic functions. In other words, there is absolutely no theoretical limitations on these functions and various models can be produced by arbitrary choices. Of course, some choices may lead to mathematical constructs that have no physical counterparts. These are usually discarded. However, many do lead to interesting constructs that may be considered true models. The ultimate ease of realizing the models that develops from their implementation on the digital computer, makes it reasonable to experiment with a variety of models based on theoretically prescribed characteristic functions. This process of experimentation is called digital simulation and will be discussed later. Ultimately, empirical characteristic functions have to be fed into the basic model to establish practical solutions.

A discussion of these functions under static and dynamic conditions follows.

4.2 THE DENSITY-FUNCTION

The apparent density of grain stored in bins is defined as the weight of grain bounded by a unit volume. Most types of grain have rather definite density values, although the density value of a particular type of grain may vary within a range of some ±10 percent.

Variations in grain density may be due to factors such as: moisture content, compactness (looseness), pre-sure, vibrations, settlements and so on. Under normal storage conditions the density value is bounded between some minimum and maximum values which may differ by 10 to 20 percent.

If not considered merely as a constant, the one-dimensional density-function is expected to be:

(1) continuous and smooth

(2) monotonically increasing with depth

(3) strictly positive

(4) bounded between $\gamma_{min} = \gamma_o$ and γ_{max}, empirical values.

The simplest function to satisfy these requirements is a linear function

$$\gamma(y) = \begin{cases} \gamma_o + \delta y & \text{for} \quad 0 \le y \le H_d \\ \gamma_{max} & \text{for} \quad y > H_d \end{cases} \tag{4.1}$$

where δ is a constant defined by

$$\delta = \frac{\gamma_{max}-\gamma_o}{H_d} .$$

If H is considerably larger than H_d, it is preferable to choose a nonlinear function. It seems reasonable to expect that the increase in density versus depth is mainly proportional to the vertical pressure. Therefore $\gamma(y)$ is assumed to increase exponentially according to Janssen-type (static) pressure, i.e.

$$\gamma(y) = \gamma_o + (\gamma_{max} - \gamma_o) [1-\exp\{-(\frac{\mu k_o}{\rho})y\}]. \quad (4.2)$$

For most practical cases in grain bins it is sufficient to assume that the density-function is a constant, γ_{ave}, or a linear function such as Eq.(4.1).

Eq.(4.2) is particularly well fitted to the density-function in silage silos.

The density-function is independent of dynamic conditions.

4.3 THE RATIO-FUNCTION

The one-dimensional ratio-function k(y) is defined as the ratio of the lateral pressure to the vertical pressure at any depth y, i.e.

$$k(y) = \frac{p(y)}{q(y)}. \quad (4.3)$$

$k(y)$ is essentially an empirical factor. In order to investigate the nature of $k(y)$ it is necessary to rely on experimental findings reported from various sources. It seems fair to summarize these findings as follows:

(1) Under static conditions the actual lateral pressure closely agree with Janssen's curves.

(2) Under dynamic conditions the actual lateral pressure may exceed Janssen's curves (by tens to hundreds percents) until they achieve a maximum at some depth above the bottom, and then decrease rapidly. The behavior is experimentally not clear in this lower region.

4.3.1 Static Conditions

Under static conditions no grain is added or removed from the system, and it is assumed that all parameters are time-independent and that the walls do not yield appreciably under the action of the lateral pressure.

Janssen (1895) who first introduced the ratio-factor, assumed that it is an empirical constant. Koenen (1896) suggested to use the theoretical "coefficient of earth pressure at rest" to represent this constant. Thus Rankine's coefficient was borrowed for use, i.e.

$$k_o = \frac{1-\sin\phi}{1+\sin\phi} \tag{4.4}$$

where ϕ is the angle of internal friction of the grain, treated as an empirical constant. The above form of k_o was adopted by most methods and is found to be sufficient for the case of static pressure. It is therefore proposed to adopt it in the present analysis also. However, there arise some important notes.

By Eq.(4.4), k_o is a function of ϕ, the angle of internal friction of grain which is defined as "the angle whose tangent equals the ratio between the shearing resistance per unit area to the corresponding normal stress in non-cohesive grain." ϕ is actually a measure of the stability of the grain. It is frequently confused with the angle of repose θ which indeed often has the same value and also serves as a measure of stability.

ϕ is only approximately equal to θ. For most materials, although not all, it is slightly larger than θ. Now, since ϕ depends on the local normal stress (pressure), it is not constant and therefore k_o is not constant. Whenever ϕ is considered constant the effect on k_o is similar to using θ. This practice is however always on the safe side, since the resulting k_o is larger than it would have come out otherwise, namely

$$k_{o\,max} = \frac{1-\sin\theta}{1+\sin\theta} . \qquad (4.5)$$

The influence of the variable $\phi(y)$ on k_o will now be investigated. First, establish an approximation to the function $\phi=\phi(y)$. ϕ is primarily dependent on the density (and the vertical pressure). If the relationship is assumed to be proportional then, qualitatively, $\phi(y)$ behaves like $\gamma(y)$, as was established experimentally by Platonow and Kowtun (1959). Using Eq.(4.2), $\phi(y)$ can be written

$$\phi(y) = \phi_o+(\phi_{max}-\phi_o)\,[1-\exp\{-(\frac{\mu k_{o\ max}}{\rho})y\}] \qquad (4.6)$$

where

ϕ_o - angle of repose, θ

$\phi_{max}= a\phi_o$, where a is an empirical coefficient depending, among other factors, on H; i.e., a=2 approximately in deep grain bins, according to the findings of Platonow and Kowtun (1959).

$k_{o\ max}$ is defined by Eq.(4.5).

Then, substituting Eq.(4.6) in Eq.(4.4), $k_o(y)$ is finally obtained in the form

$$k_o(y) = \frac{1-\sin[\theta(2-\exp\{-\frac{\mu(1-\sin\theta)}{\rho(1+\sin\theta)}y\})]}{1+\sin[\theta(2-\exp\{-\frac{\mu(1-\sin\theta)}{\rho(1+\sin\theta}y\})]} \qquad (4.7)$$

and $k_o(y)$ is thereby expressible as a function of y, involving θ, μ and ρ as parameters. The behavior of the function $k_o(y)$ may be seen by using the trigonometric identity

$$\frac{1-\sin\phi}{1+\sin\phi} = \tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right). \tag{4.8}$$

Let $(\frac{\pi}{4} - \frac{\phi}{2}) = \psi$ and examine ψ. By Eq. (4.6), $\frac{\phi}{2}$ is increasing versus y, and therefore ψ is decreasing. The tangent of the decreasing argument ψ, where $0 < \psi < \frac{\pi}{4}$, is decreasing versus y and its square is even more so. It follows that $k_o(y)$ is a decreasing function of y, convex toward the y-axis. Such behavior can well be represented by the appropriate choice of a hyperbola such as the type used in section 3.3.2.

4.3.2 Dynamic Conditions

It is generally accepted that dynamic effects occur during charging and especially discharging. The discussion below is confined to discharging alone.

As mentioned before, the determination of k(y) should be done experimentally. In fact, under dynamic conditions the ratio-function must be time-dependent, k(y,t), and this further complicates the picture. In the absence of real experimental dynamic data the discussion below is somewhat speculative.

Suppose that as a result of a discharging operation, the walls deflect to the extent that the contact between the grain and the walls is reduced to a minimum. At this particular instant two main phenomena may occur:

(1) Since the friction effect approaches zero, the vertical pressure, and therefore the lateral pressure as well, rapidly increase. If the deflection of the walls is sufficiently large, so that the system can be looked upon as if the walls were completely removed (for a moment), the column of grain tends to collapse and transform into the characteristic pile (see the discussion in the section on topological models).

(2) At the same instant the walls, being released from pressure, tend to return. The assumption is that the maximum lateral pressure develops when the returning walls and the nearly collapsing grain interact.

"Dynamic conditions" will be regarded as conditions under which the two following requirements are satisfied:

(1) The lateral pressure is, qualitatively at least, in agreement with the experimental findings described in the previous section.

(2) The walls deflect appreciably.

Further studies, both digital simulations and experimentation, should be made to establish reliable $k(y,t)$ functions under various dynamic conditions. In Chapter 7 a theoretical attempt is made to characterize a dynamic ratio-function.

4.4 THE FRICTION-FUNCTION

The friction-function $\lambda(y)$ is defined as the ratio of the frictional stress at the wall to the lateral pressure, at any depth, namely

$$\lambda(y) = \frac{s(y)}{p(y)} . \tag{4.9}$$

The coefficient of friction of the grain on the wall μ is basically dependent on the type of the grain, the material of the wall, and the degree of smoothness (roughness) of the wall surface. Other factors such as: variations of the moisture content of the grain, and the motion of the grain, may affect the coefficient of friction. Nevertheless, this factor is normally considered as a constant.

The wall-friction parameter λ was introduced in order to accommodate variations in the friction effect, either as a function of the depth coordinate y, or independent of y, but as a variable parameter depending on other factors. Therefore,

$$\lambda = \nu\mu \tag{4.10}$$

where ν is either (i) a constant, (ii) a function of y, or (iii) a variable, but independent of y. The flexibility in the definition of ν allows to take into account many possible variations in the friction effect. ν is made to be bounded between 0 and 1, such that λ is bounded between 0 and μ.

4.4.1 Static Conditions

The friction effect attains its maximum influence under static conditions. Therefore let $\nu=1$ such that

$$\lambda = \lambda_{max} = \mu. \tag{4.11}$$

4.4.2 Dynamic Conditions

Since ν is bounded between 0 and 1, it may prove useful to express it in the form of a trigo-ometric function, i.e. the sine function. Let

$$\nu(y) = \sin cy \tag{4.12}$$

where c is a parameter.

This representation may be appropriate especially for a case where the wall vibrates under dynamic conditions. When the wall vibrates, it is likely to deflect in the form of a sine curve. The mode of the vibrations is not known (involved in c), but can be estimated for simple cases. It is assumed that the wall is much more flexible than the mass of the grain is, and that the grain mass cannot continuously take on the form of the deflecting wall. Therefore the grain is in very loose contact with the wall, or in no contact at all, along the intervals where the wall is convex outwards. Therefore define $\nu(y)$ as follows:

$$\nu(y) = \begin{cases} \sin cy & \text{for } \sin cy > 0 \\ 0 & \text{for } \sin cy \leq 0 \end{cases} \tag{4.13}$$

From the nature of the sine curve and definition (4.13), it is evident that regardless of the mode of the vibrations $\nu(y)$ vanishes intermittently over a cumulative length of one-half of the total depth of the bin.

Since the friction effect is important as a cumulative factor, it is not necessary to find the exact mode of the deflections. The mode can be taken for convenience as the arbitrary interval Δy, according to which the numerical calculations are carried out. This choice produces a $\nu(y)$ that vanishes at every other interval and varies between 0 and 1 according to a sine curve at every other interval. For numerical calculations of sufficiently small Δy, $\nu(y)$ approaches the form of a step-function. Therefore it can be written as follows,

$$\nu(y) = \begin{cases} \text{ave. } \sin cy & \text{for } \sin cy > 0 \\ 0 & \text{for } \sin cy \leq 0 \end{cases} \tag{4.14}$$

where

$$\text{ave. } \sin cy = \frac{2}{\pi}.$$

Therefore, under "simple" dynamic conditions and a choice of sufficiently small increment, $\nu(y)$ may be chosen to a step-function of the form

$$\nu(y) = \begin{cases} \frac{2}{\pi} & \text{for i odd integer} \\ 0 & \text{for i even integer} \end{cases} \tag{4.15}$$

where i is related to Eq. (3.28), i.e. $i=y/\Delta y$. Accordingly the friction-function can be expressed as a step-function of the form

$$\lambda(y) \approx \begin{cases} \frac{2}{\pi}\mu & \text{for i odd integer} \\ 0 & \text{for i even integer .} \end{cases} \tag{4.16}$$

4.4.3 Practical Recommendations

Eq. (4.16) may suggest to choose a constant friction-function which is the average of the intermittent values 0 and $\frac{2}{\pi}\mu$, namely

$$\lambda = \frac{1}{\pi}\mu \simeq \frac{1}{3}\mu \ .$$

The choice of constants smaller than this is justified for cases where the assumption of simple sinusidal deflection is not valid. In any case of uncertainty it is advisable to choose smaller values for the parameter ν, in-

cluding the possibility of a zero value. When $\nu=0$, the friction effect is completely neglected and thus the lateral and bottom pressures attain maximum values. However, to calculate the vertical load on the walls, $\lambda_{max}=\mu$ should be used.

In summary, there may exist three cases of significant difference (among many possible intermediate cases):

(1) Static conditions. ν is constant = 1. The friction parameter attains its upper bound, $\lambda_{max}=\mu$.

(2) "Simple" dynamic conditions. $\nu(y)$ is a step-function of the form given by Eq.(4.15); or, more practically, ν can be taken as an averaged constant, namely, $\nu = \frac{1}{3}$ and therefore $\lambda = \frac{1}{3}\mu$.

(3) Complex dynamic conditions. $\mu = 0$ and $\lambda = 0$.

For design purposes the following is suggested:

(1) For rigid structures with special discharging devices (such as a central perforated pipe) use

$$\lambda = \lambda_{max} = \mu.$$

(2) For rigid structures without special discharging devices, or for semi-rigid structures with such devices, use

$$\lambda = \frac{1}{3}\mu.$$

(3) For non-rigid structures with hazardous possibilities for dynamic effects, use

$\lambda \to 0$, or $\lambda = 0$.

Notes. The following are general examples of different types of grain bins that may occur in practice, with respect to the degree of "rigidity" as used in the above recommendations. For more accurate determination of any specific design case, a complete structural analysis should be made.

Rigid structures are structures such as clusters of monolithic reinforced concrete bins with rigid connections to a rigid slab foundation, based on a high-quality soil and with rigid connections at the tops.

Semi-rigid structures are single, exceptionally tall reinforced concrete bins; clusters of steel bins.

Non-rigid structures are single, tall, steel bins with a non-rigid connection to the foundation, based on an inferior soil and subjected to frequent gusts.

Proper discharging devices are such as those reported in the literature and tested in operation in many countries. However, the design and installation of such devices, do not necessarily guarantee that the dynamic effects will be eliminated. Therefore, a conservative choice of the values of the parameter ν is encouraged.

CHAPTER 5

NUMERICAL AND COMPUTER MODELS

5.1 NUMERICAL MODELS BASED ON THE ANALYTIC MODEL

The numerical models of this category are based on the procedure described in section 3.2. The model is written first in the form of Eqs. (3.18) and (3.19), $\beta(y)$ being specified by the particular $k(y)$ and $\lambda(y)$ chosen for the model. Then Eq. (3.18) is programmed into the FUNCTION subprogram named FUN. The control program calls the fourth-order Runge-Kutta integration subroutine RK2 (identical with subroutine RUNKUT given in section 3.2). Subroutine RK2 returns the discrete solution for the vertical pressure $q(y)$. The control program calculates the lateral pressure $p(y)$ by multiplying $q(y)$ by $k(y)$.

5.1.1 Examples and Results

(1) Example No. 1. The characteristic functions are:

Density-function:

$$\gamma = 45.0 \text{ lb/cu.ft.} \quad \text{(constant).}$$

Ratio-function:

$$k(y) = \frac{2k_o}{1+0.06y}$$

where,

$k_o = 0.333.$

Friction-function:

$\lambda = \mu = 0.450$ (constant).

The system solved is:

$$\begin{cases} \dfrac{dq}{dy} \doteq \gamma - \dfrac{\mu}{R} \cdot \dfrac{2k_o}{1+0.06y} \cdot q \\ q(0) = 0 \end{cases}$$

Geometry:

$H = 100.00$ ft.

$R = 4.99$ ft.

$\Delta y = 1.00$ ft.

The final results give the lateral pressure, p(y), in tabular form in both kg/m^2 and lb/sq.ft. The same results are also plotted at 2.0 ft. and 1.0 ft. intervals.

NUMERICAL SOLUTION OF THE
ANALYTIC MODEL

Example No. 1

```
      DIMENSION A(500),XX(500)
      EXTERNAL FUN
    1 FORMAT(3F10.0,2I5)
    2 FORMAT(1H1,7X,44HSOLUTION OF DY/DX=FUN(X,Y) BY RK2 SUBROUTINE//
     11H ,10X ,2HH=,F7.3,2X,3HX0=,F7.3,2X,3HY0=,F7.3//
     21H,12X,1HX,18X,4HY(X)//)
    3 FORMAT(1H,10X,F5.2,10X,E15.8)
   10 READ(5,1) X0,Y0,H,JNT,IENT
   20 WRITE(6,2) H,X0,Y0
      CALL RK2(FUN,H,X0,Y0,JNT,IENT,A)
      STEP=FLOAT(JNT)*H
      X=X0
      DO 30 I=1,IENT
      XX(I)=X
      X=X+STEP
      A(I)=A(I)*((0.666)/(1.0+0.06*XX(I)))
      B=A(I)*4.882
      WRITE(6,751) X,B
  751 FORMAT(10X,F10.5,10X,F15.6)
   30 WRITE(6,3) X,A(I)
      CALL AMAX(A,IENT,VALMAX,ISUB)
      CALL AMIN(A,IENT,VALMIN,ISUB)
      CALL PLOT(1,IENT,1,VALMAX,VALMIN,XX,A)
      CALL PLOT(0,IENT,1,VALMAX,VALMIN,XX,A)
      VALMAX=2.0*VALMAX
      CALL PLOT(1,IENT,1,VALMAX,VALMIN,XX,A)
      CALL PLOT(0,IENT,1,VALMAX,VALMIN,XX,A)
      WRITE(6,750) IENT
  750 FORMAT(' TOTAL NO. OF INCREMENTS=',I5)
   40 CONTINUE
      RETURN
      END
```

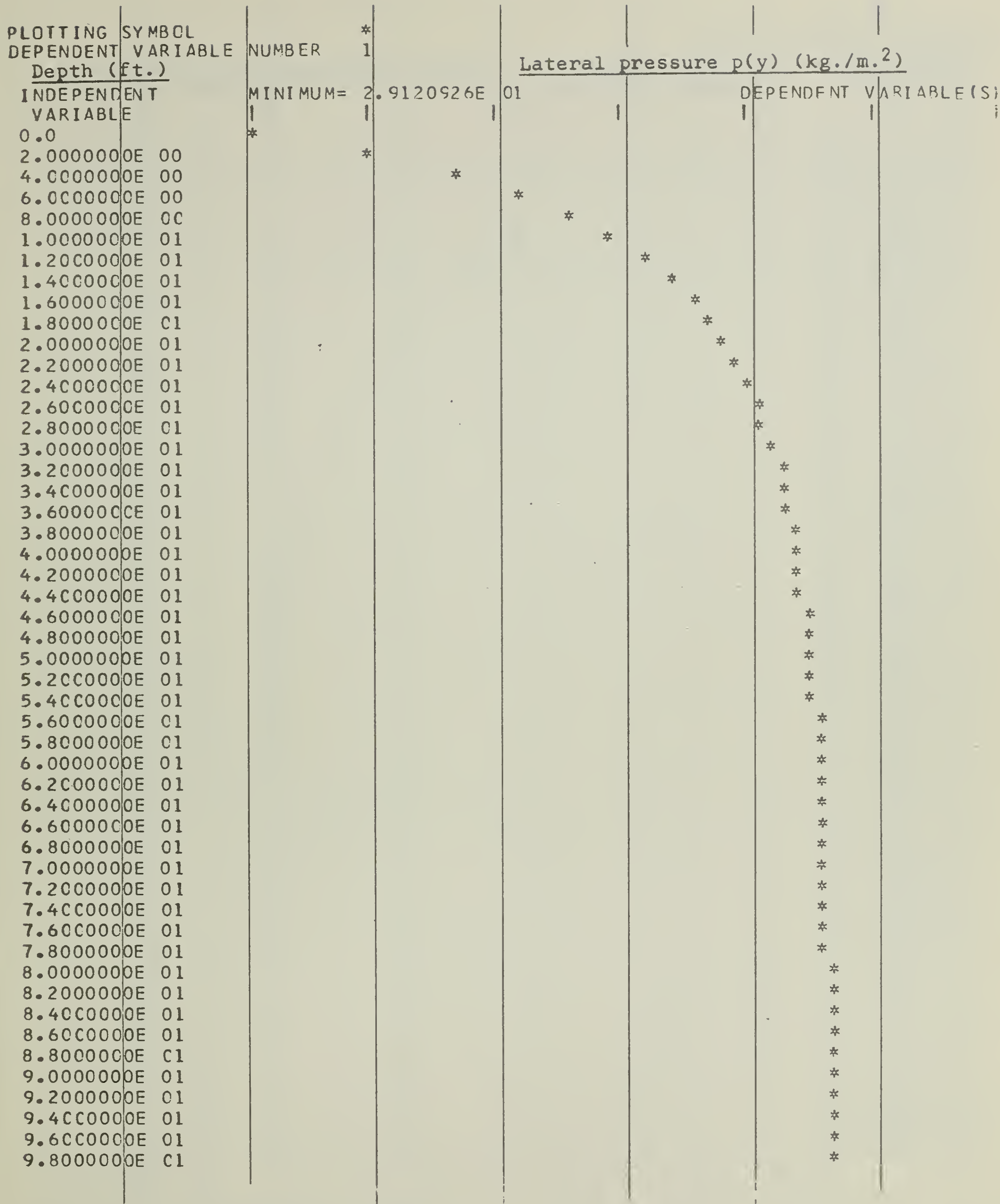
```
      FUNCTION FUN(X,Y)
      FUN=45.0-(0.450/4.99)*((0.666)/(1.+(0.06*X)))*Y
      RETURN
      END
```

SOLUTION OF DY/DX=FUN(X,Y) BY RK2 SUBROUTINE

H= 1.000 X0= 0.0 Y0= 0.0

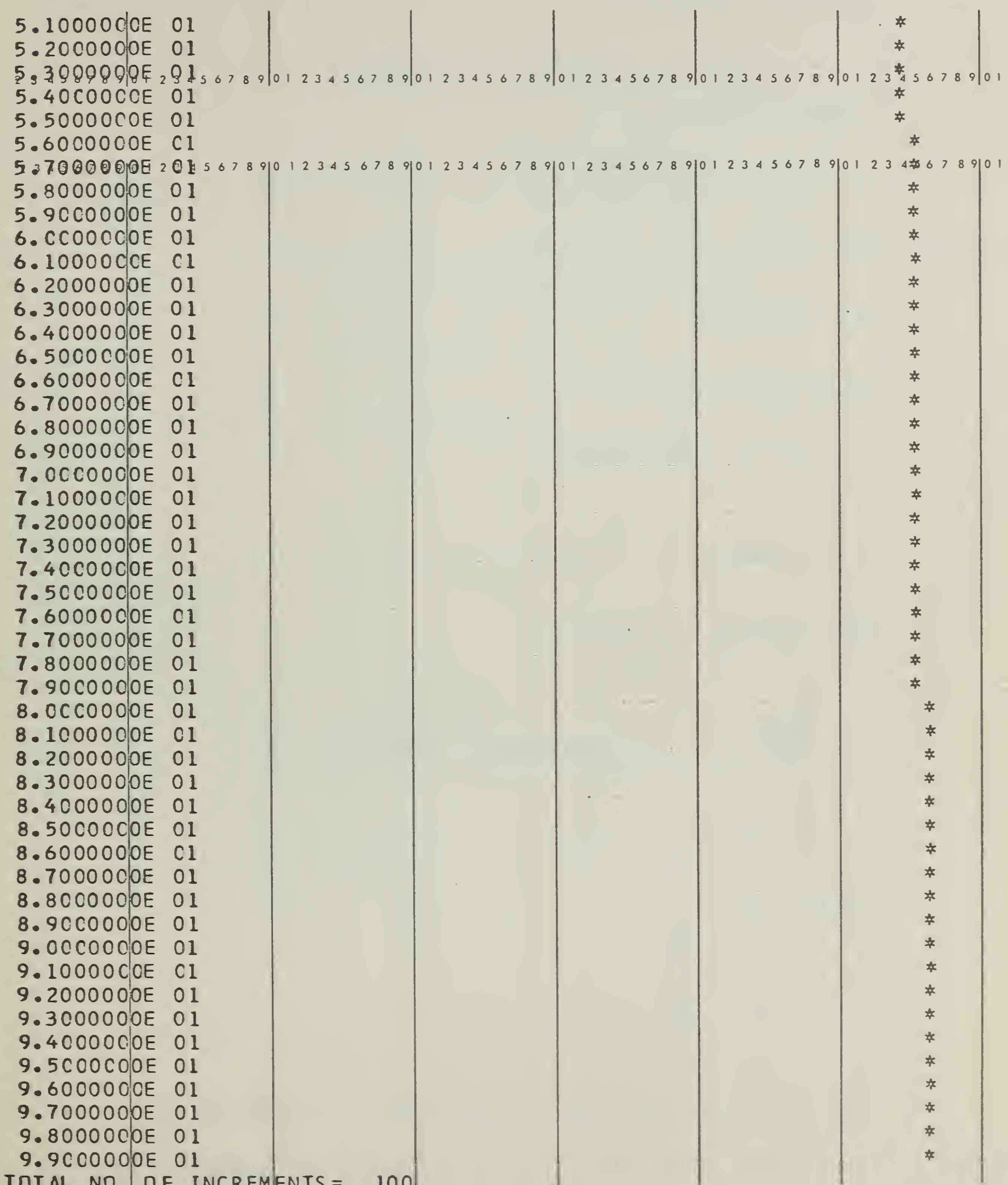
Depth (ft.)	Lateral pressure p(y) (kg./m.2 and lb./sq. ft.)	Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)
1.00000	142.168350	26.00000	1057.677002
1.00	0.29120926E 02	26.00	0.21664833E 03
2.00000	261.259766	27.00000	1065.708984
2.00	0.53514923E 02	27.00	0.21829353E 03
3.00000	361.990967	28.00000	1073.186279
3.00	0.74148117E 02	28.00	0.21982516E 03
4.00000	447.935303	29.00000	1080.159180
4.00	0.91752426E 02	29.00	0.22125345E 03
5.00000	521.837158	30.00000	1086.670166
5.00	0.10689005E 03	30.00	0.22258713E 03
6.00000	585.834961	31.00000	1092.760010
6.00	0.11999901E 03	31.00	0.22383450E 03
7.00000	641.612305	32.00000	1098.462402
7.00	0.13142410E 03	32.00	0.22500256E 03
8.00000	690.511963	33.00000	1103.810303
8.00	0.14144043E 03	33.00	0.22609798E 03
9.00000	733.612793	34.00000	1108.830566
9.00	0.15026892E 03	34.00	0.22712634E 03
10.00000	771.791016	35.00000	1113.550537
10.00	0.15808913E 03	35.00	0.22809312E 03
11.00000	805.764404	36.00000	1117.992187
11.00	0.16504803E 03	36.00	0.22900293E 03
12.00000	836.123291	37.00000	1122.177246
12.00	0.17126656E 03	37.00	0.22986020E 03
13.00000	863.359619	38.00000	1126.125244
13.00	0.17684550E 03	38.00	0.23066884E 03
14.00000	887.883545	39.00000	1129.852539
14.00	0.18186885E 03	39.00	0.23143231E 03
15.00000	910.041504	40.00000	1133.375244
15.00	0.18640755E 03	40.00	0.23215392E 03
16.00000	930.125000	41.00000	1136.708008
16.00	0.19052132E 03	41.00	0.23283655E 03
17.00000	948.383789	42.00000	1139.864258
17.00	0.19426135E 03	42.00	0.23348306E 03
18.00000	965.029297	43.00000	1142.854980
18.00	0.19767093E 03	43.00	0.23409567E 03
19.00000	980.245361	44.00000	1145.692383
19.00	0.20078767E 03	44.00	0.23467688E 03
20.00000	994.189209	45.00000	1148.386475
20.00	0.20364384E 03	45.00	0.23522870E 03
21.00000	1006.997070	46.00000	1150.946045
21.00	0.20626735E 03	46.00	0.23575299E 03
22.00000	1018.788818	47.00000	1153.380371
22.00	0.20868268E 03	47.00	0.23625166E 03
23.00000	1029.666504	48.00000	1155.696533
23.00	0.21091081E 03	48.00	0.23672606E 03
24.00000	1039.722412	49.00000	1157.902588
24.00	0.21297060E 03	49.00	0.23717796E 03
25.00000	1049.035645	50.00000	1160.005127
25.00	0.21487827E 03	50.00	0.23760860E 03

Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)	Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)
51.00000	1162.010010	76.00000	1192.177002
51.00	0.23801930E 03	76.00	0.24419852E 03
52.00000	1163.923584	77.00000	1192.883057
52.00	0.23841124E 03	77.00	0.24434312E 03
53.00000	1165.750732	78.00000	1193.565674
53.00	0.23878549E 03	78.00	0.24448296E 03
54.00000	1167.497070	79.00000	1194.225342
54.00	0.23914323E 03	79.00	0.24461809E 03
55.00000	1169.166748	80.00000	1194.863037
55.00	0.23948524E 03	80.00	0.24474872E 03
56.00000	1170.763916	81.00000	1195.480469
56.00	0.23981238E 03	81.00	0.24487515E 03
57.00000	1172.293213	82.00000	1196.077148
57.00	0.24012561E 03	82.00	0.24499741E 03
58.00000	1173.757324	83.00000	1196.654297
58.00	0.24042554E 03	83.00	0.24511563E 03
59.00000	1175.160645	84.00000	1197.213867
59.00	0.24071300E 03	84.00	0.24523022E 03
60.00000	1176.506836	85.00000	1197.755371
60.00	0.24098872E 03	85.00	0.24534117E 03
61.00000	1177.797852	86.00000	1198.279297
61.00	0.24125316E 03	86.00	0.24544847E 03
62.00000	1179.036621	87.00000	1198.787842
62.00	0.24150691E 03	87.00	0.24555264E 03
63.00000	1180.226563	88.00000	1199.279785
63.00	0.24175067E 03	88.00	0.24565343E 03
64.00000	1181.369873	89.00000	1199.757324
64.00	0.24198486E 03	89.00	0.24575122E 03
65.00000	1182.468994	90.00000	1200.220459
65.00	0.24220999E 03	90.00	0.24584608E 03
66.00000	1183.525635	91.00000	1200.669922
66.00	0.24242641E 03	91.00	0.24593814E 03
67.00000	1184.542725	92.00000	1201.105225
67.00	0.24263475E 03	92.00	0.24602733E 03
68.00000	1185.521240	93.00000	1201.528076
68.00	0.24283516E 03	93.00	0.24611395E 03
69.00000	1186.463623	94.00000	1201.938477
69.00	0.24302820E 03	94.00	0.24619798E 03
70.00000	1187.371582	95.00000	1202.337158
70.00	0.24321422E 03	95.00	0.24627966E 03
71.00000	1188.246338	96.00000	1202.723877
71.00	0.24339340E 03	96.00	0.24635889E 03
72.00000	1189.089844	97.00000	1203.099854
72.00	0.24356613E 03	97.00	0.24643590E 03
73.00000	1189.903076	98.00000	1203.464844
73.00	0.24373274E 03	98.00	0.24651067E 03
74.00000	1190.688232	99.00000	1203.819824
74.00	0.24389355E 03	99.00	0.24658337E 03
75.00000	1191.445557	100.00000	1204.164795
75.00	0.24404869E 03	*****	0.24665404E 03


PLOTTING SYMBOL *
DEPENDENT VARIABLE NUMBER 1
Depth (ft.)
Lateral pressure p(y) (kg./m.^{2})
INDEPENDENT VARIABLE
MINIMUM= 2.9120926E 01
DEPENDENT VARIABLE(S)
0.0
2.0000000E 00
4.0000000E 00
6.0000000E 00
8.0000000E 00
1.0000000E 01
1.2000000E 01
1.4000000E 01
1.6000000E 01
1.8000000E 01
2.0000000E 01
2.2000000E 01
2.4000000E 01
2.6000000E 01
2.8000000E 01
3.0000000E 01
3.2000000E 01
3.4000000E 01
3.6000000E 01
3.8000000E 01
4.0000000E 01
4.2000000E 01
4.4000000E 01
4.6000000E 01
4.8000000E 01
5.0000000E 01
5.2000000E 01
5.4000000E 01
5.6000000E 01
5.8000000E 01
6.0000000E 01
6.2000000E 01
6.4000000E 01
6.6000000E 01
6.8000000E 01
7.0000000E 01
7.2000000E 01
7.4000000E 01
7.6000000E 01
7.8000000E 01
8.0000000E 01
8.2000000E 01
8.4000000E 01
8.6000000E 01
8.8000000E 01
9.0000000E 01
9.2000000E 01
9.4000000E 01
9.6000000E 01
9.8000000E 01

```
PLOTTING SYMBOL                    *
DEPENDENT VARIABLE NUMBER          1
  Depth (ft.)                                   Lateral pressure p(y) (kg./m.^2)
 INDEPENDENT            MINIMUM= 2.91209 26E 01                DEPENDENT VARIABLE(S)
  VARIABLE              I          I          I          I          I          I
 0.0                 *
 1.00000000E 00           *
 2.00000000E 00                *
 3.00000000E 00                    *
 4.00000000E 00                       *
 5.00000000E 00                           *
 6.00000000E 00                             *
 7.00000000E 00                               *
 8.00000000E 00                                 *
 9.00000000E 00                                   *
 1.00000000E 01                                    *
 1.10000000E 01                                      *
 1.20000000E 01                                        *
 1.30000000E 01                                         *
 1.40000000E 01                                          *
 1.50000000E 01                                           *
 1.60000000E 01                                            *
 1.70000000E 01                                            *
 1.80000000E 01                                             *
 1.90000000E 01                                              *
 2.00000000E 01                                              *
 2.10000000E 01                                               *
 2.20000000E 01                                               *
 2.30000000E 01                                                *
 2.40000000E 01                                                *
 2.50000000E 01                                                *
 2.60000000E 01                                                 *
 2.70000000E 01                                                 *
 2.80000000E 01                                                 *
 2.90000000E 01                                                  *
 3.00000000E 01                                                  *
 3.10000000E 01                                                  *
 3.20000000E 01                                                   *
 3.30000000E 01                                                   *
 3.40000000E 01                                                   *
 3.50000000E 01                                                   *
 3.60000000E 01                                                   *
 3.70000000E 01                                                   *
 3.80000000E 01                                                    *
 3.90000000E 01                                                    *
 4.00000000E 01                                                    *
 4.10000000E 01                                                    *
 4.20000000E 01                                                    *
 4.30000000E 01                                                    *
 4.40000000E 01                                                    *
 4.50000000E 01                                                     *
 4.60000000E 01                                                     *
 4.70000000E 01                                                     *
 4.80000000E 01                                                     *
 4.90000000E 01                                                     *
 5.00000000E 01                                                     *
```

Depth (ft.) Lateral pressure p(y) (kg./m.^{2})



TOTAL NO. OF INCREMENTS= 100

(2) Example No. 2. The characteristic functions are:

Density-function:

$$\gamma = 45.0 \text{ lb/cu.ft.} \quad \text{(constant)}.$$

Ratio-function:

$$k(y) = \frac{2k_o}{1+0.06y}$$

where,

$$k_o = 0.333.$$

Friction-function:

$$\lambda(y) = \mu|\cos(y)|$$

where,

$$\mu = 0.425.$$

The system solved is:

$$\begin{cases} \dfrac{dq}{dy} = \gamma - \dfrac{\mu}{R} \cdot |\cos(y)| \cdot \dfrac{2k_o}{1+0.06y} \cdot q \\ \\ q(0) = 0 \end{cases}$$

Geometry:

$H = 100.00$ ft.

$R = 4.99$ ft.

$\Delta y = 1.00$ ft.

The results show that the lateral pressure is larger than in the previous case, basically because of the periodic decrease in the value of the friction-function.

NUMERICAL SOLUTION OF THE ANALYTIC MODEL

Example No. 2

```
      DIMENSION A(500),XX(500)
      EXTERNAL FUN
    1 FORMAT(3F10.0,2I5)
    2 FORMAT(1H1,7X,44HSOLUTION OF DY/DX=FUN(X,Y) BY RK2 SUBROUTINE//
     11H,10X,2HH=,F7.3,2X,3HX0=,F7.3,2X,3HY0=,F7.3//
     21H,12X,1HX,18X,4HY(X)///)
    3 FORMAT(1H,10X,F5.2,10X,E15.8)
   10 READ(5,1) X0,Y0,H,JNT,IENT
   20 WRITE(6,2) H,X0,Y0
      CALL RK2(FUN,H,X0,Y0,JNT,IENT,A)
      STEP=FLOAT(JNT)*H
      X=X0
      DO 30 I=1,IENT
      XX(I)=X
      X=X+STEP
      A(I)=A(I)*((0.666)/(1.0+0.06*XX(I)))
      B=A(I)*4.882
      WRITE(5,751) X,B
  751 FORMAT(10X,F10.5,10X,F15.6)
   30 WRITE(6,3) X,A(I)
      CALL AMAX(A,IENT,VALMAX,ISUB)
      CALL AMIN(A,IENT,VALMIN,ISUB)
      CALL PLOP(1,IENT,1,VALMAX,VALMIN,XX,A)
      CALL PLOP(0,IENT,1,VALMAX,VALMIN,XX,A)
      VALMAX=2.0*VALMAX
      CALL PLOP(1,IENT,1,VALMAX,VALMIN,XX,A)
      CALL PLOP(0,IENT,1,VALMAX,VALMIN,XX,A)
      WRITE(6,750) IENT
  750 FORMAT(' TOTAL NO. OF INCREMENTS=',I5)
   40 CONTINUE
      CALL EXIT
      END
```

```
      FUNCTION FUN(X,Y)
      FLAMDA = 0.425 * ABS ( COS(X) )
      FUN = 45.0 - (FLAMDA/4.99)*((0.666)/(1.+(0.06*X)))*Y
      RETURN
      END
```

SOLUTION OF DY/DX=FUN(X,Y) BY RK2 SUBROUTINE

H= 1.000 X0= 0.0 Y0= 0.0

Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)	Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq.ft.)
1.00000	143.296326	26.00000	1208.455322
1.00	0.29351974E 02	26.00	0.24753285E 03
2.00000	271.109863	27.00000	1229.616455
2.00	0.55532578E 02	27.00	0.25186740E 03
3.00000	374.987793	28.00000	1239.354980
3.00	0.76810303E 02	28.00	0.25386218E 03
4.00000	462.814453	29.00000	1242.205078
4.00	0.94800201E 02	29.00	0.25444595E 03
5.00000	551.970947	30.00000	1253.906250
5.00	0.11306248E 03	30.00	0.25786694E 03
6.00000	622.187500	31.00000	1269.505859
6.00	0.12744522E 03	31.00	0.26003809E 03
7.00000	678.049561	32.00000	1270.459229
7.00	0.13888770E 03	32.00	0.26023340E 03
8.00000	742.216064	33.00000	1283.483154
8.00	0.15203116E 03	33.00	0.26290112E 03
9.00000	794.481689	34.00000	1295.376953
9.00	0.16273697E 03	34.00	0.26533740E 03
10.00000	830.998291	35.00000	1295.245850
10.00	0.17021680E 03	35.00	0.26531055E 03
11.00000	878.069336	36.00000	1304.248291
11.00	0.17985855E 03	36.00	0.26715454E 03
12.00000	920.593262	37.00000	1316.739502
12.00	0.18856891E 03	37.00	0.26971313E 03
13.00000	945.094971	38.00000	1316.234131
13.00	0.19358768E 03	38.00	0.26960962E 03
14.00000	978.622314	39.00000	1321.830078
14.00	0.20045523E 03	39.00	0.27075586E 03
15.00000	1014.670654	40.00000	1335.061279
15.00	0.20783916E 03	40.00	0.27346606E 03
16.00000	1031.829346	41.00000	1334.742920
16.00	0.21135385E 03	41.00	0.27340088E 03
17.00000	1055.226318	42.00000	1337.577393
17.00	0.21614633E 03	42.00	0.27398145E 03
18.00000	1087.448486	43.00000	1351.691650
18.00	0.22274655E 03	43.00	0.27687256E 03
19.00000	1100.272461	44.00000	1351.996826
19.00	0.22537331E 03	44.00	0.27693506E 03
20.00000	1116.038574	45.00000	1352.640381
20.00	0.22860275E 03	45.00	0.27706669E 03
21.00000	1146.063477	46.00000	1366.250732
21.00	0.23475290E 03	46.00	0.27985474E 03
22.00000	1156.542725	47.00000	1367.549505
22.00	0.23689941E 03	47.00	0.28012085E 03
23.00000	1166.546143	48.00000	1366.551025
23.00	0.23894846E 03	48.00	0.27991626E 03
24.00000	1193.109619	49.00000	1377.490234
24.00	0.24438953E 03	49.00	0.28215698E 03
25.00000	1202.682861	50.00000	1380.113525
25.00	0.24635045E 03	50.00	0.28269434E 03

Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)	Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)
51.00000	1378.020752	76.00000	1437.665527
51.00	0.28226562E 03	76.00	0.29448291E 03
52.00000	1386.647705	77.00000	1442.021729
52.00	0.28403271E 03	77.00	0.29537524E 03
53.00000	1390.816895	78.00000	1445.651123
53.00	0.28488672E 03	78.00	0.29611865E 03
54.00000	1388.107666	79.00000	1442.452891
54.00	0.28433179E 03	79.00	0.29546558E 03
55.00000	1394.753662	80.00000	1445.115967
55.00	0.28569312E 03	80.00	0.29600903E 03
56.00000	1400.606934	81.00000	1449.770264
56.00	0.28689209E 03	81.00	0.29695240E 03
57.00000	1397.689209	82.00000	1446.301270
57.00	0.28629443E 03	82.00	0.29635425E 03
58.00000	1401.959961	83.00000	1447.860840
58.00	0.28716919E 03	83.00	0.29657129E 03
59.00000	1408.879883	84.00000	1453.459229
59.00	0.28858667E 03	84.00	0.29771802E 03
60.00000	1406.139893	85.00000	1450.960937
60.00	0.28802539E 03	85.00	0.29720630E 03
61.00000	1408.285156	86.00000	1450.702430
61.00	0.28846484E 03	86.00	0.29715088E 03
62.00000	1416.230469	87.00000	1457.267334
62.00	0.29009229E 03	87.00	0.29849805E 03
63.00000	1413.994385	88.00000	1455.449707
63.00	0.28963428E 03	88.00	0.29812573E 03
64.00000	1414.384277	89.00000	1454.107666
64.00	0.28971411E 03	89.00	0.29785083E 03
65.00000	1423.400879	90.00000	1461.056396
65.00	0.29156104E 03	90.00	0.29927417E 03
66.00000	1421.937256	91.00000	1460.107666
66.00	0.29126123E 03	91.00	0.29907983E 03
67.00000	1420.922852	92.00000	1457.953857
67.00	0.29105347E 03	92.00	0.29863857E 03
68.00000	1430.074219	93.00000	1463.521240
68.00	0.29292798E 03	93.00	0.29977905E 03
69.00000	1429.617676	94.00000	1463.615479
69.00	0.29283447E 03	94.00	0.29979834E 03
70.00000	1427.547363	95.00000	1460.927734
70.00	0.29241040E 03	95.00	0.29924780E 03
71.00000	1434.890625	96.00000	1465.259033
71.00	0.29391455E 03	96.00	0.30013501E 03
72.00000	1435.658203	97.00000	1466.527100
72.00	0.29407178E 03	97.00	0.30039478E 03
73.00000	1432.895503	98.00000	1463.564209
73.00	0.29350586E 03	98.00	0.29977784E 03
74.00000	1438.641602	99.00000	1466.810791
74.00	0.29468286E 03	99.00	0.30045288E 03
75.00000	1440.789307	100.00000	1469.342529
75.00	0.29512280E 03	*****	0.30097144E 03

```
PLOTTING SYMBOL                         *
DEPENDENT VARIABLE NUMBER               1
  Depth (ft.)                                         Lateral pressure p(y) (kg./m.2)
 INDEPENDENT          MINIMUM= 2.9351974E 01                      DEPENDENT VARIABLE(S)
  VARIABLE            |          |          |          |          |          |          |
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 4.0000000E 00                          *
 6.0000000E 00                              *
 8.0000000E 00                                   *
 1.0000000E 01                                      *
 1.2000000E 01                                        *
 1.4000000E 01                                          *
 1.6000000E 01                                            *
 1.8000000E 01                                             *
 2.0000000E 01                                               *
 2.2000000E 01                                                *
 2.4000000E 01                                                 *
 2.6000000E 01                                                  *
 2.8000000E 01                                                  *
 3.0000000E 01                                                   *
 3.2000000E 01                                                    *
 3.4000000E 01                                                    *
 3.6000000E 01                                                     *
 3.8000000E 01                                                     *
 4.0000000E 01                                                      *
 4.2000000E 01                                                      *
 4.4000000E 01                                                      *
 4.6000000E 01                                                       *
 4.8000000E 01                                                       *
 5.0000000E 01                                                       *
 5.2000000E 01                                                        *
 5.4000000E 01                                                        *
 5.6000000E 01                                                        *
 5.8000000E 01                                                        *
 6.0000000E 01                                                        *
 6.2000000E 01                                                         *
 6.4000000E 01                                                         *
 6.6000000E 01                                                         *
 6.8000000E 01                                                         *
 7.0000000E 01                                                         *
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 8.8000000E 01                                                          *
 9.0000000E 01                                                          *
 9.2000000E 01                                                          *
 9.4000000E 01                                                          *
 9.6000000E 01                                                          *
 9.8000000E 01                                                          *
```
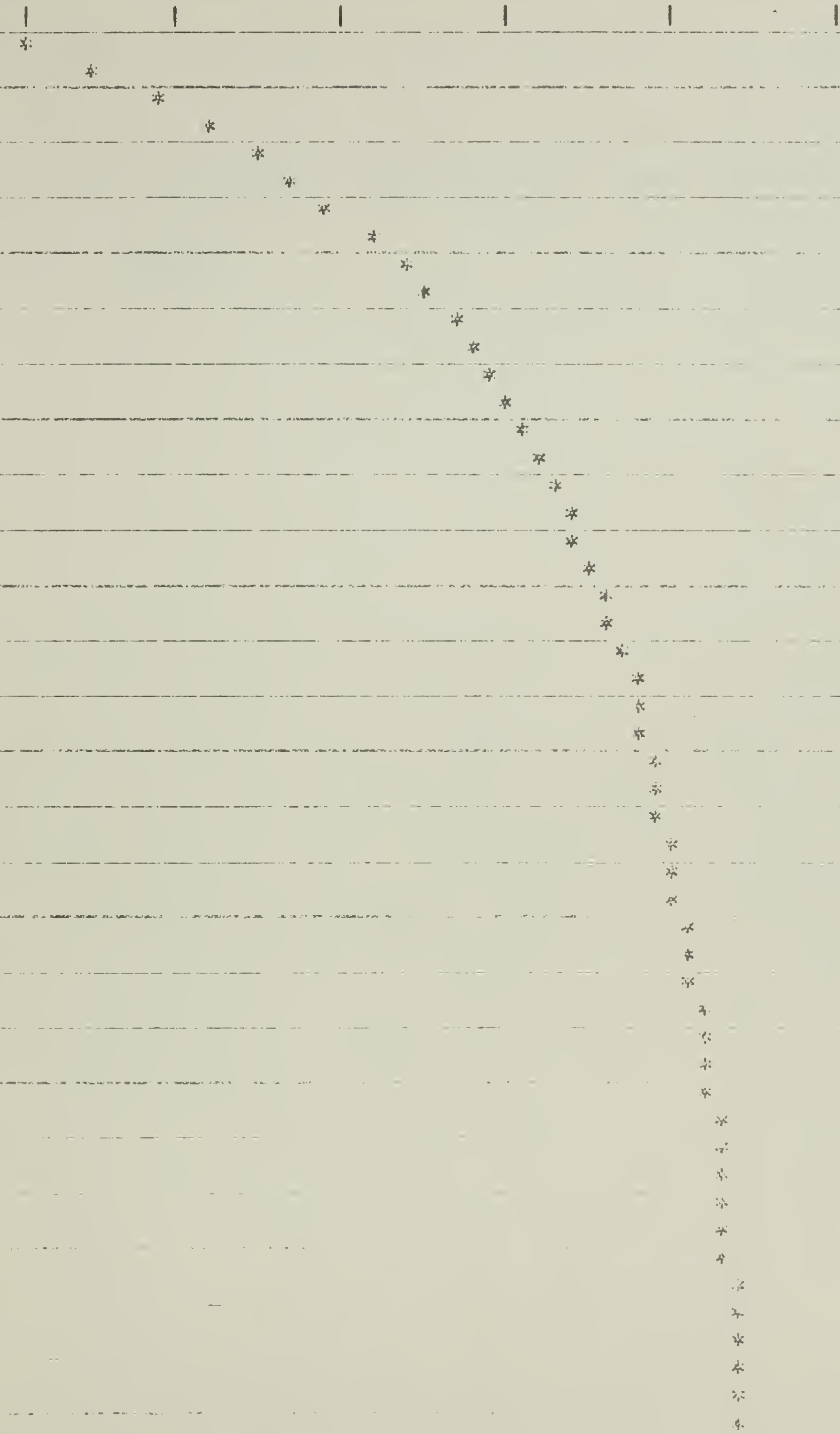

PLOTTING SYMBOL *

DEPENDENT VARIABLE NUMBER 1

Depth (ft.) Lateral pressure p(y) (kg./m.^{2})

INDEPENDENT VARIABLE MINIMUM= 2.9351974E 01 DEPENDENT VARIABLE(S)



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4.0000000E 00
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7.0000000E 00
8.0000000E 00
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4.7000000E 01
4.8000000E 01
4.9000000E 01
5.0000000E 01

Depth (ft.)	Lateral pressure $p(y)$ (kg./m.^{2})
5.1000000E 01	*
5.2000000E 01	*
5.3000000E 01	*
5.4000000E 01	*
5.5000000E 01	*
5.6000000E 01	*
5.7000000E 01	*
5.8000000E 01	*
5.9000000E 01	*
6.0000000E 01	*
6.1000000E 01	*
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6.4000000E 01	*
6.5000000E 01	*
6.6000000E 01	*
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6.8000000E 01	*
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7.8000000E 01	*
7.9000000E 01	*
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8.8000000E 01	*
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9.2000000E 01	*
9.3000000E 01	*
9.4000000E 01	*
9.5000000E 01	*
9.6000000E 01	*
9.7000000E 01	*
9.8000000E 01	*
9.9000000E 01	*

TOTAL NO. OF INCREMENTS= 100

(3) Example No. 3. This example is identical with example No. 2, except for the value of the friction coefficient μ, i.e., $\mu = 0.850$. The large value assumed for μ and the periodic behavior of $\lambda(y)$ produce pressure fluctuations in the solution. For details, see the plots accompanying this example.

NUMERICAL SOLUTION OF THE
ANALYTIC MODEL

Example No. 3

```
      DIMENSION A(500),XX(500)
      EXTERNAL FUN
    1 FORMAT(3F10.0,2I5)
    2 FORMAT(1H1,7X,44HSOLUTION OF DY/DX=FUN(X,Y) BY RK2 SUBROUTINE//
     11H,10X,2HH=,F7.3,2X,3HX0=,F7.3,2X,3HY0=,F7.3//
     21H,12X,1HX,18X,4HY(X)//)
    3 FORMAT(1H,10X,F5.2,10X,E15.8)
   10 READ(5,1) X0,Y0,H,JNT,IENT
   20 WRITE(6,2) H,X0,Y0
      CALL RK2(FUN,H,X0,Y0,JNT,IENT,A)
      STEP=FLOAT(JNT)*H
      X=X0
      DO 30 I=1,IENT
      XX(I)=X
      X=X+STEP
      A(I)=A(I)*((0.666)/(1.0+0.06*XX(I)))
      B=A(I)*4.882
      WRITE(6,751) X,B
  751 FORMAT(10X,F10.5,10X,F15.6)
   30 WRITE(6,3) X,A(I)
      CALL AMAX(A,IENT,VALMAX,ISUB)
      CALL AMIN(A,IENT,VALMIN,ISUB)
      CALL PLOP(1,IENT,1,VALMAX,VALMIN,XX,A)
      CALL PLOP(0,IENT,1,VALMAX,VALMIN,XX,A)
      VALMAX=2.0*VALMAX
      CALL PLOP(1,IENT,1,VALMAX,VALMIN,XX,A)
      CALL PLOP(0,IENT,1,VALMAX,VALMIN,XX,A)
      WRITE(6,750) IENT
  750 FORMAT(' TOTAL NO. OF INCREMENTS=',I5)
   40 CONTINUE
      CALL EXIT
      END
```

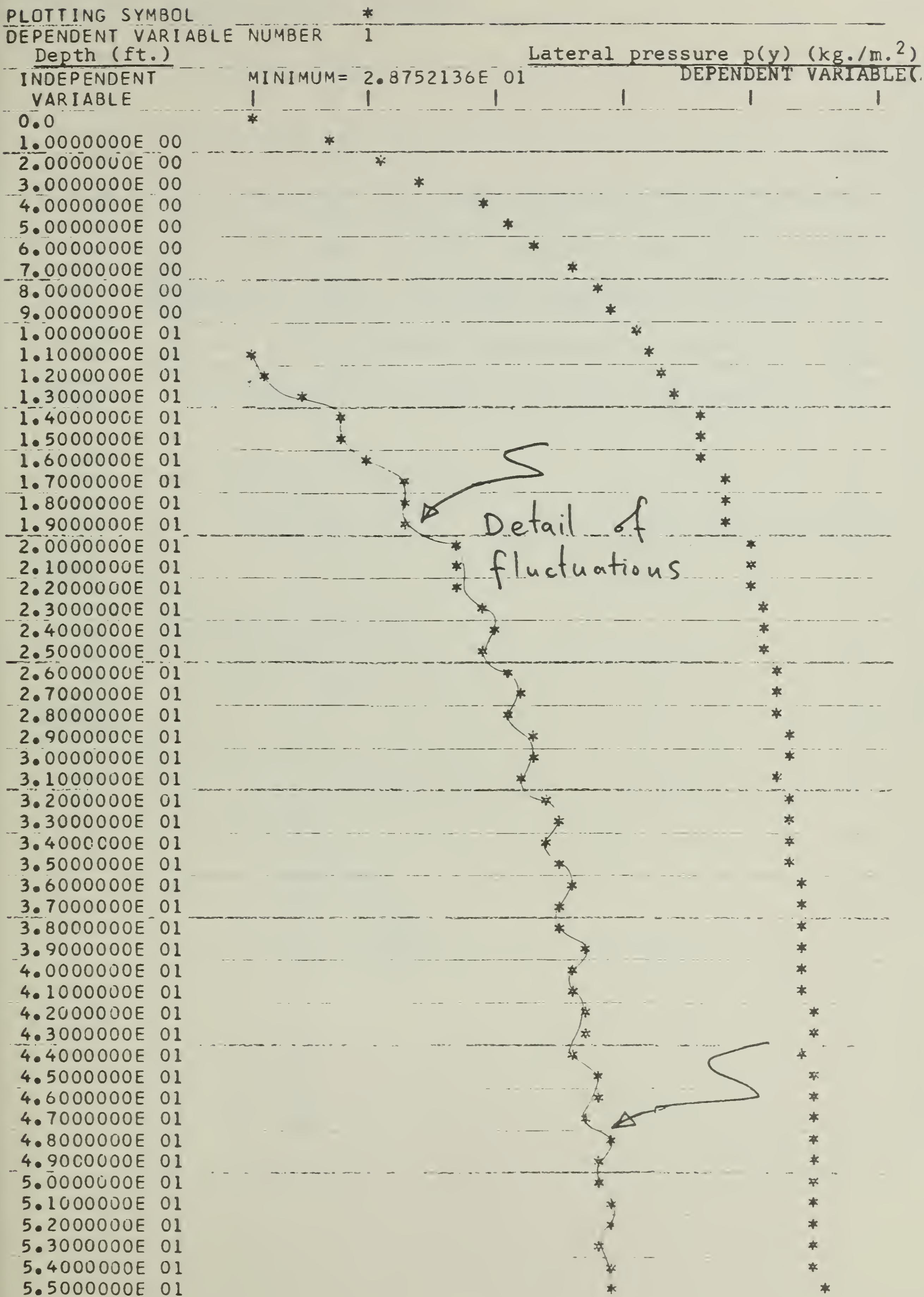
```
      FUNCTION FUN(X,Y)
      FLAMDA = 0.850 * ABS ( COS(X) )
      FUN = 45.0 - (FLAMDA/4.99)*((0.666)/(1.+(0.06*X)))*Y
      RETURN
      END
```

SOLUTION OF DY/DX=FUN(X,Y) BY RK2 SUBROUTINE

H= 1.000 X0= 0.0 Y0= 0.0

Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)	Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)
1.00000	140.367920	26.00000	982.937500
1.00	0.28752136E 02	26.00	0.20133911E 03
2.00000	266.310791	27.00000	1004.559814
2.00	0.54549530E 02	27.00	0.20576811E 03
3.00000	358.965576	28.00000	1008.114502
3.00	0.73528427E 02	28.00	0.20649625E 03
4.00000	432.407959	29.00000	1001.284912
4.00	0.88571915E 02	29.00	0.20509731E 03
5.00000	517.428223	30.00000	1017.449219
5.00	0.10598697E 03	30.00	0.20840833E 03
6.00000	574.696533	31.00000	1024.171631
6.00	0.11771745E 03	31.00	0.20978531E 03
7.00000	612.612305	32.00000	1015.980713
7.00	0.12548390E 03	32.00	0.20810753E 03
8.00000	671.320557	33.00000	1027.667725
8.00	0.13750935E 03	33.00	0.21050142E 03
9.00000	713.176514	34.00000	1037.849365
9.00	0.14608290E 03	34.00	0.21258694E 03
10.00000	731.459717	35.00000	1029.277100
10.00	0.14982790E 03	35.00	0.21083107E 03
11.00000	772.277344	36.00000	1035.721924
11.00	0.15818875E 03	36.00	0.21215120E 03
12.00000	808.063721	37.00000	1047.955566
12.00	0.16551903E 03	37.00	0.21465706E 03
13.00000	815.428223	38.00000	1039.910156
13.00	0.16702754E 03	38.00	0.21300908E 03
14.00000	841.225830	39.00000	1041.942871
14.00	0.17231177E 03	39.00	0.21342546E 03
15.00000	873.391113	40.00000	1056.232178
15.00	0.17890031E 03	40.00	0.21635239E 03
16.00000	875.291748	41.00000	1049.401123
16.00	0.17928960E 03	41.00	0.21495316E 03
17.00000	889.916992	42.00000	1047.908936
17.00	0.18228537E 03	42.00	0.21464749E 03
18.00000	920.904297	43.00000	1064.295410
18.00	0.18863260E 03	43.00	0.21800401E 03
19.00000	920.644043	44.00000	1059.184326
19.00	0.18857933E 03	44.00	0.21695709E 03
20.00000	926.958984	45.00000	1054.938477
20.00	0.18987282E 03	45.00	0.21608737E 03
21.00000	958.094727	46.00000	1071.118408
21.00	0.19625046E 03	46.00	0.21940157E 03
22.00000	957.718262	47.00000	1068.181152
22.00	0.19617337E 03	47.00	0.21879996E 03
23.00000	957.856445	48.00000	1061.945068
23.00	0.19620168E 03	48.00	0.21752255E 03
24.00000	986.110107	49.00000	1074.458252
24.00	0.20198897E 03	49.00	0.22008569E 03
25.00000	987.125000	50.00000	1074.126953
25.00	0.20219685E 03	50.00	0.22001785E 03

Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)	Depth (ft.)	Lateral pressure p(y) (kg./m.^{2} and lb./sq. ft.)
51.00000	1066.703369	76.00000	1089.028320
51.00	0.21849722E 03	76.00	0.22307013E 03
52.00000	1076.064209	77.00000	1093.831543
52.00	0.22041464E 03	77.00	0.22405399E 03
53.00000	1078.594971	78.00000	1097.539062
53.00	0.22093306E 03	78.00	0.22481343E 03
54.00000	1070.662598	79.00000	1090.976807
54.00	0.21930823E 03	79.00	0.22346924E 03
55.00000	1077.346924	80.00000	1093.332764
55.00	0.22067741E 03	80.00	0.22395186E 03
56.00000	1082.862061	81.00000	1098.727295
56.00	0.22180708E 03	81.00	0.22505682E 03
57.00000	1074.988281	82.00000	1092.633301
57.00	0.22019427E 03	82.00	0.22380858E 03
58.00000	1078.346436	83.00000	1092.704590
58.00	0.22088213E 03	83.00	0.22382318E 03
59.00000	1085.831299	84.00000	1099.642090
59.00	0.22241528E 03	84.00	0.22524420E 03
60.00000	1078.565918	85.00000	1094.379639
60.00	0.22092711E 03	85.00	0.22416629E 03
61.00000	1078.958008	86.00000	1092.557373
61.00	0.22100739E 03	86.00	0.22379300E 03
62.00000	1088.293457	87.00000	1101.069092
62.00	0.22291963E 03	87.00	0.22553650E 03
63.00000	1082.092285	88.00000	1096.937988
63.00	0.22164941E 03	88.00	0.22469034E 03
64.00000	1080.061035	89.00000	1093.598145
64.00	0.22123334E 03	89.00	0.22400621E 03
65.00000	1091.265381	90.00000	1102.756104
65.00	0.22352837E 03	90.00	0.22588205E 03
66.00000	1086.489014	91.00000	1100.022705
66.00	0.22254999E 03	91.00	0.22532217E 03
67.00000	1082.536865	92.00000	1095.542725
67.00	0.22174049E 03	92.00	0.22440454E 03
68.00000	1094.144043	93.00000	1102.700684
68.00	0.22411800E 03	93.00	0.22587071E 03
69.00000	1091.114746	94.00000	1101.627197
69.00	0.22349751E 03	94.00	0.22565083E 03
70.00000	1085.745850	95.00000	1096.427979
70.00	0.22239777E 03	95.00	0.22458585E 03
71.00000	1094.772705	96.00000	1101.804932
71.00	0.22424678E 03	96.00	0.22568724E 03
72.00000	1093.807617	97.00000	1102.580078
72.00	0.22404910E 03	97.00	0.22584601E 03
73.00000	1087.571045	98.00000	1097.046875
73.00	0.22277162E 03	98.00	0.22471262E 03
74.00000	1094.332275	99.00000	1100.867920
74.00	0.22415657E 03	99.00	0.22549532E 03
75.00000	1095.642578	100.00000	1103.613037
75.00	0.22442496E 03	*****	0.22605762E 03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5.4000000E 01
5.5000000E 01
Detail of fluctuations

Depth (ft.) | Lateral pressure p(y) (kg./m.^{2})

5.6000000E 01
5.7000000E 01
5.8000000E 01
5.9000000E 01
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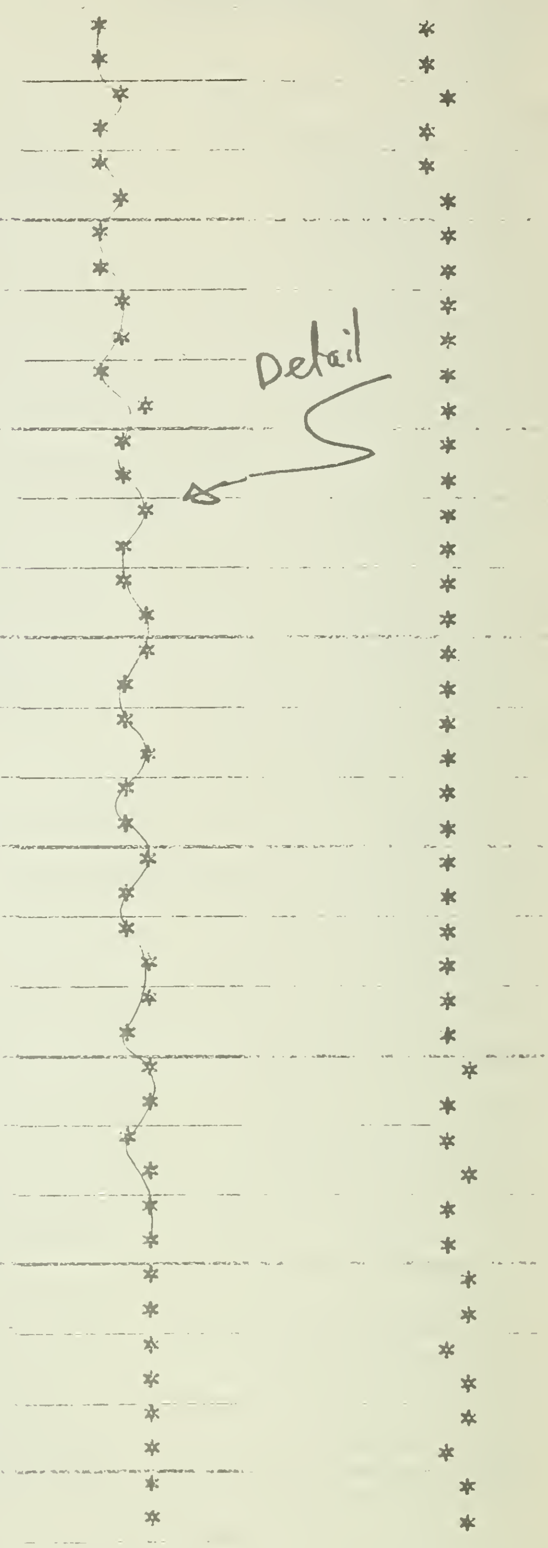
7.1000000E 01
7.2000000E 01
7.3000000E 01
7.4000000E 01
7.5000000E 01
7.6000000E 01
7.7000000E 01
7.8000000E 01
7.9000000E 01
8.0000000E 01
8.1000000E 01
8.2000000E 01
8.3000000E 01
8.4000000E 01
8.5000000E 01
8.6000000E 01
8.7000000E 01
8.8000000E 01
8.9000000E 01
9.0000000E 01
9.1000000E 01
9.2000000E 01
9.3000000E 01
9.4000000E 01
9.5000000E 01
9.6000000E 01
9.7000000E 01
9.8000000E 01
9.9000000E 01



TOTAL NO. OF INCREMENTS= 100

5.2 COMPUTERIZED GRAPHIC REPRESENTATION OF THE ANALYTIC MODEL

The models represented in the form of Eq. (3.18) describe three-dimensional surfaces over the y-q plane. Graphic visualization of these surfaces may aid in the model building process. Surfaces describing various analytic models were plotted automatically by a Calcomp plotter. The program listings are given in Appendix B.

5.2.1 Examples and Results

(1) Example No. 1. The characteristic functions are:

Density-function:

$$\gamma(y) = \gamma_o + \delta y$$

where,

$\gamma_o = 45.0$ lb/cu.ft.

$\delta = 0.06$

Ratio-function:

$$k(y) = \frac{2k_o}{1+\left(\frac{\mu k_o}{R}\right)y}$$

where,

$k_o = 0.333$

$\mu = 0.425$

$R = 5.0$ ft.

Friction-function:

$$\lambda = 0.425 \quad \text{(constant)}.$$

The surface plotted is:

$$\frac{dq}{dy} = \gamma(y) - \frac{\lambda}{R} \cdot k(y)q(y)$$

$$0 \leq y \leq 120 \text{ ft.}$$

$$0 \leq q \leq 400 \text{ lb/sq.ft.}$$

The plotted surface can be given in different orientations, registered by the coordinate system at the upper right corner of each plot. Three representative plots of this surface are given in Figs. 5.1, 5.2, 5.3.

(2) <u>Example No. 2</u>. The model is the same as in example No. 1, except that

$$\lambda(y) = \mu|\cos(y)|.$$

Three orientations of the resulting surfaces are given in Figs. 5.4, 5.5, 5.6.

(3) <u>Example No. 3</u>. The model is the same as in example No. 1, except that

$$\lambda(y) = \mu|\sin(y)|.$$

Three orientations of the resulting surfaces are given in Figs. 5.7, 5.8, 5.9.

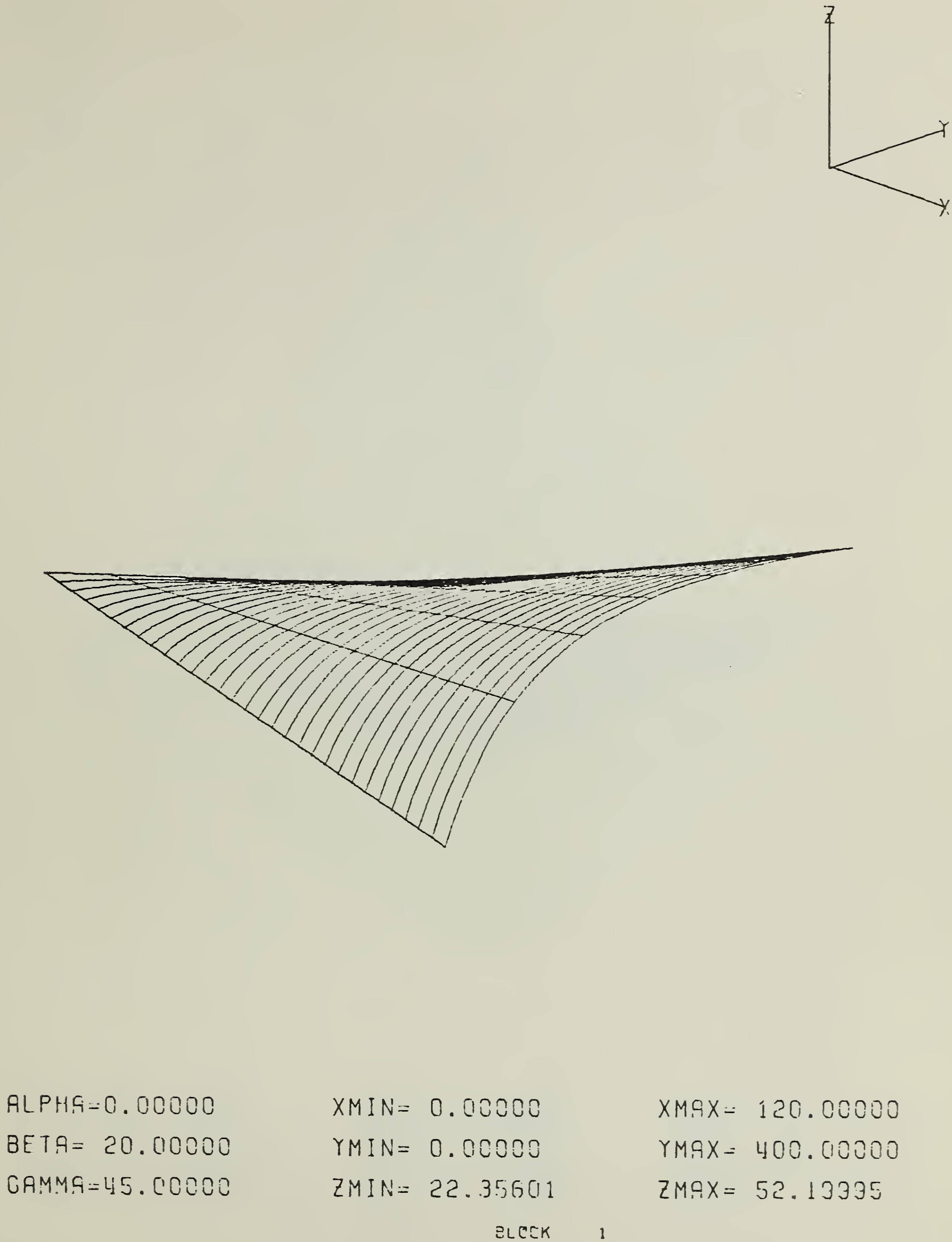
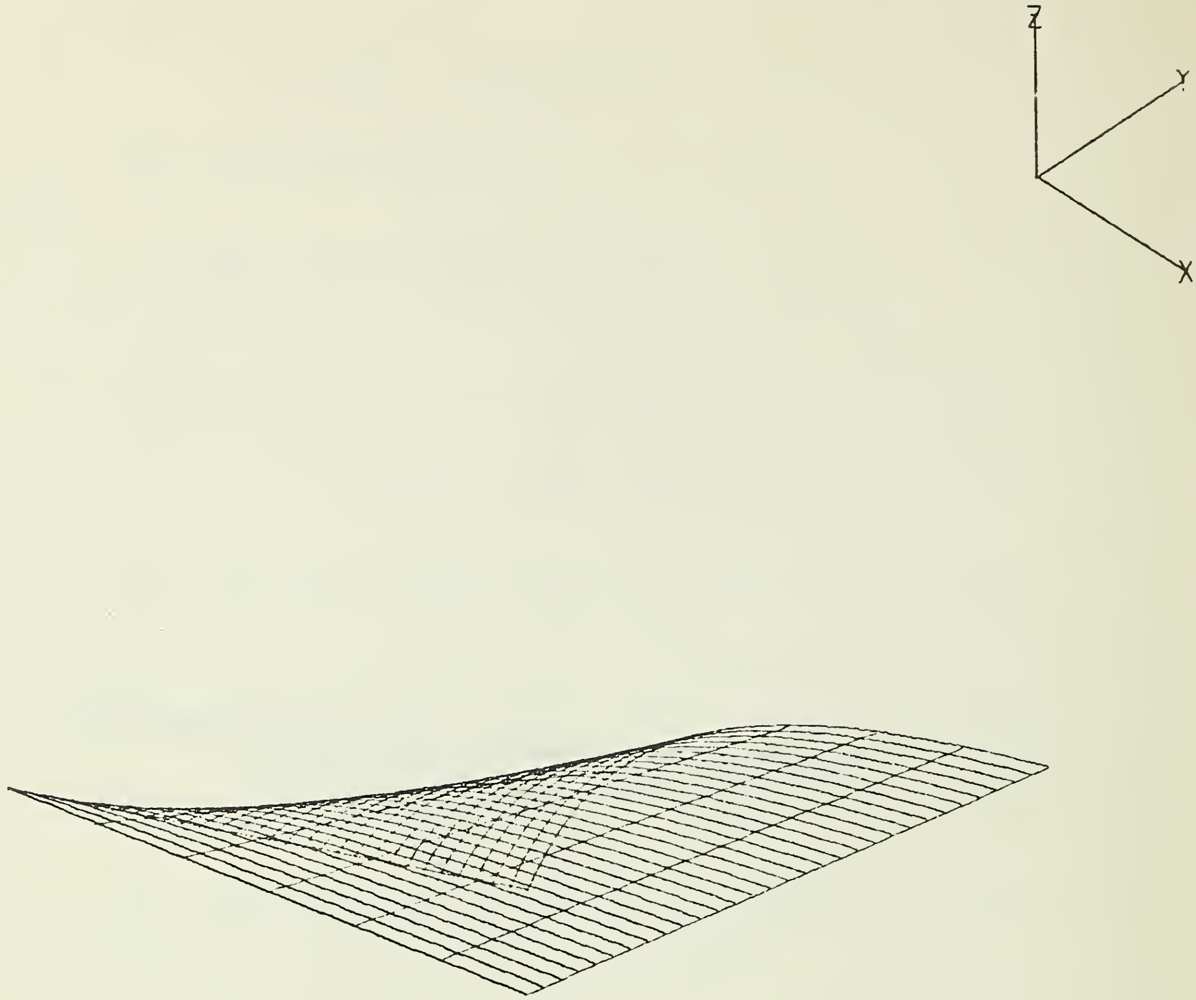

Z
Y
X
ALPHA=0.00000
BETA= 20.00000
GAMMA=45.00000
XMIN= 0.00000
YMIN= 0.00000
ZMIN= 22.35601
XMAX= 120.00000
YMAX= 400.00000
ZMAX= 52.13395
BLOCK 1

Figure 5.1



ALPHA=0.00000	XMIN= 0.00000	XMAX= 120.00000
BETA= 40.00000	YMIN= 0.00000	YMAX= 400.00000
GAMMA=45.00000	ZMIN= 22.35601	ZMAX= 52.19995

BLOCK 2

Figure 5.2

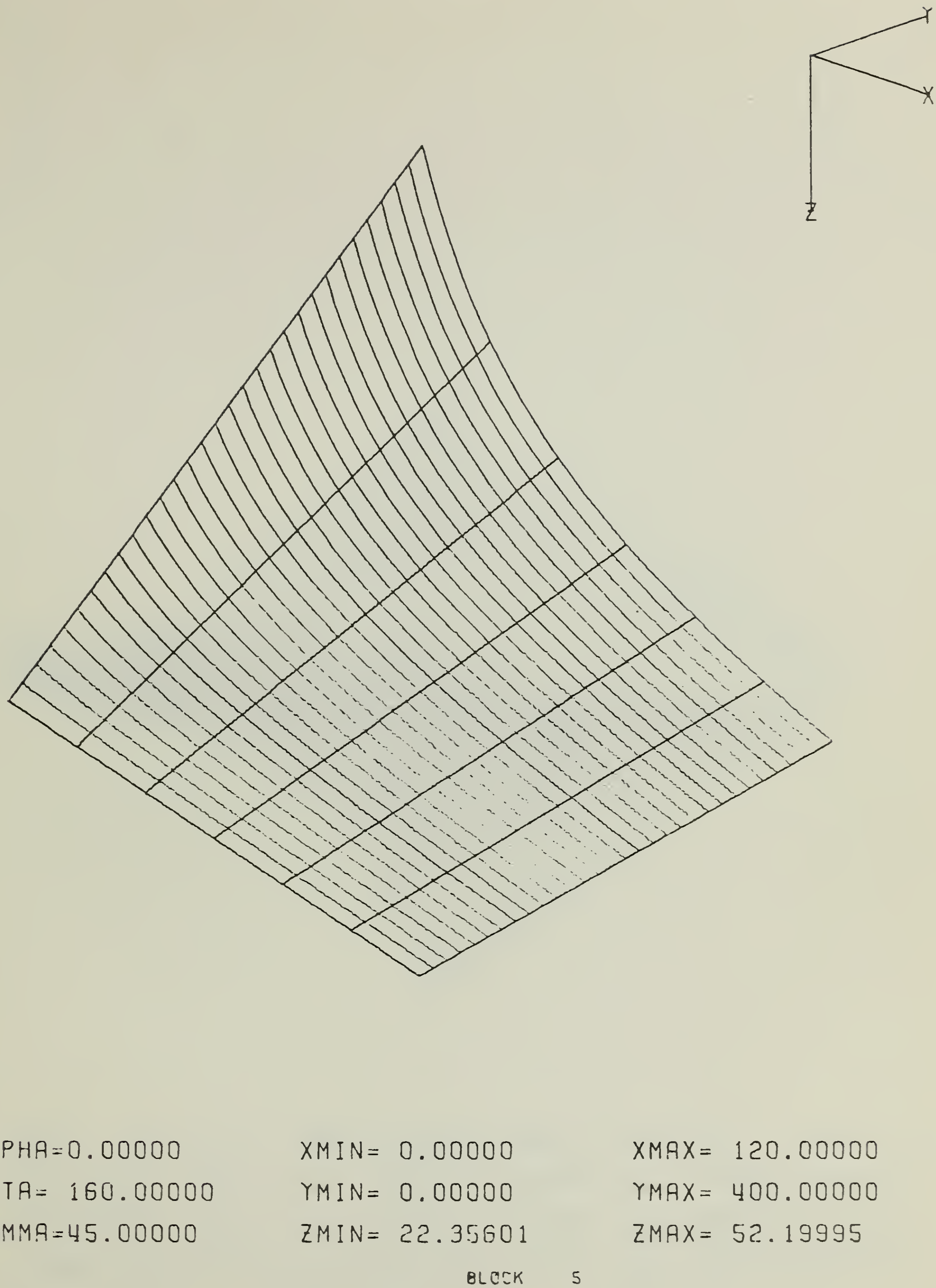
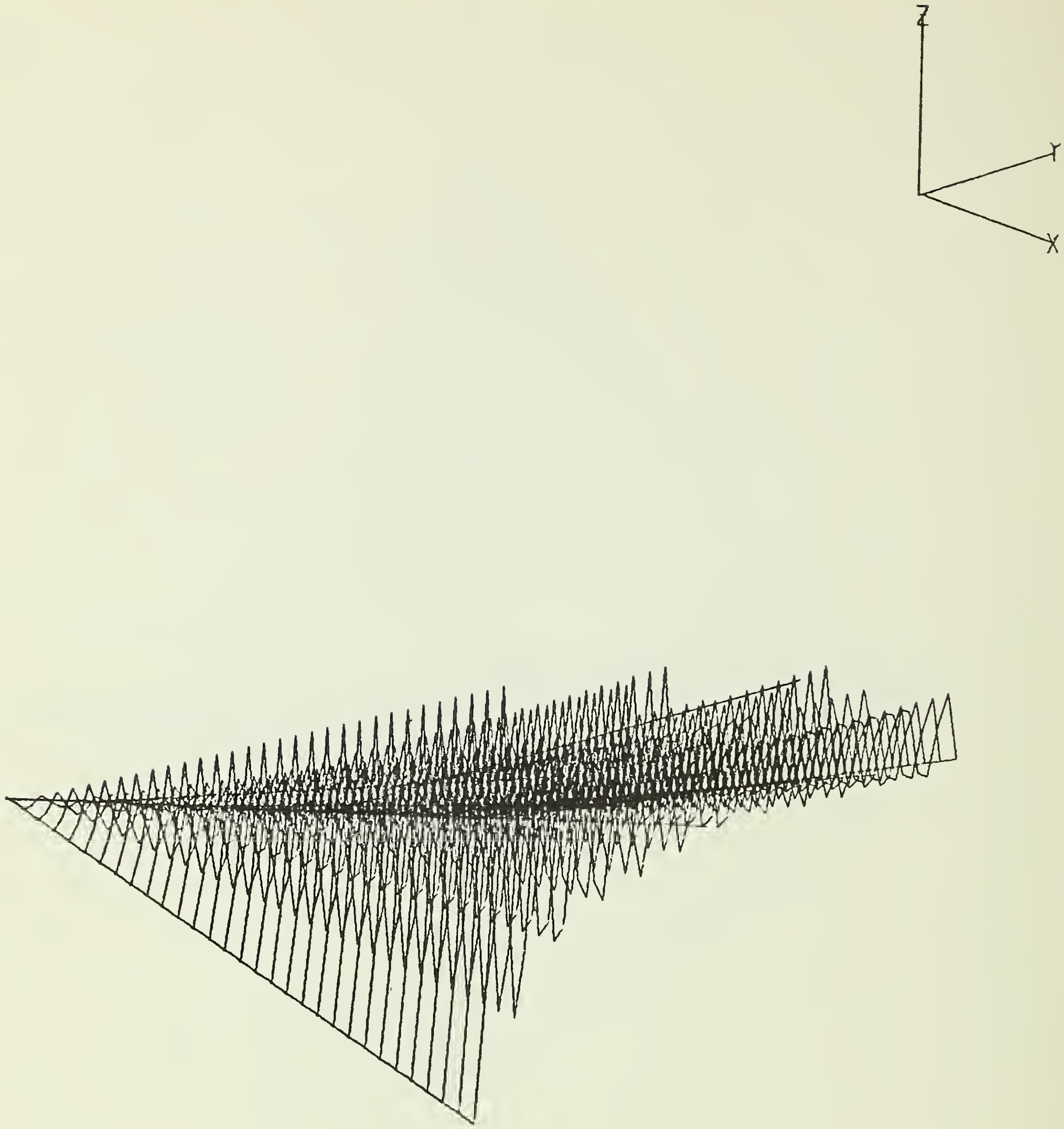

Y
X
Z
ALPHA=0.00000
BETA= 160.00000
GAMMA=45.00000
XMIN= 0.00000
YMIN= 0.00000
ZMIN= 22.35601
XMAX= 120.00000
YMAX= 400.00000
ZMAX= 52.19995
BLOCK 5

Figure 5.3

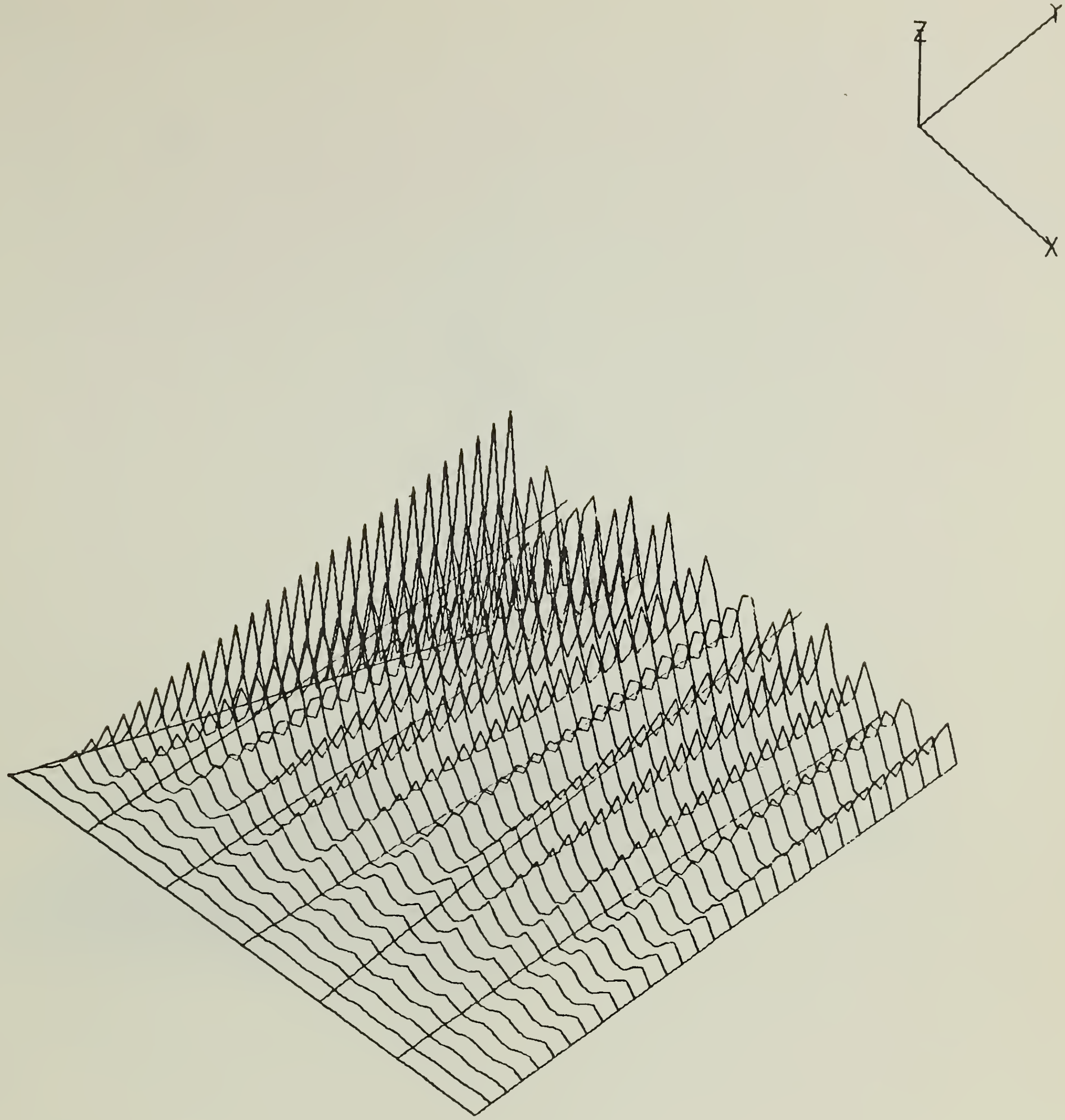


Z = 45. + .06 * X - .056 * ABS(COS(X)) * Y / (1 + .0283 * X)

ALPHA=0.00000	XMIN= 0.00000	XMAX= 120.00000
BETA= 20.00000	YMIN= 0.00000	YMAX= 400.00000
GAMMA=45.00000	ZMIN= 22.35601	ZMAX= 52.19995

BLOCK 1

Figure 5.4

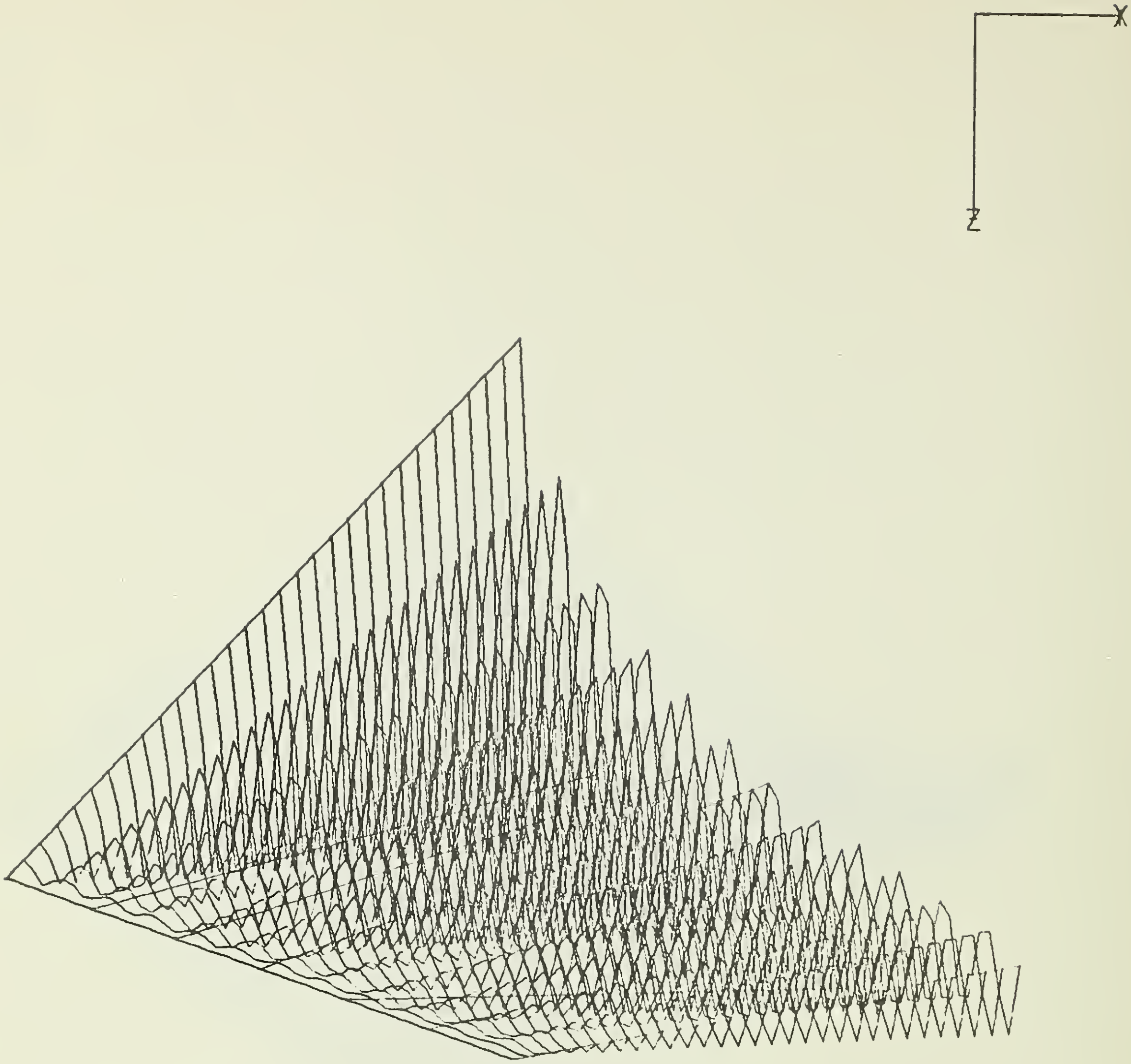


Z = 45. + .06 * X - .056 * ABS(COS(X)) * Y / (1 + 0283 * X)

ALPHA=0.00000	XMIN= 0.00000	XMAX= 120.00000
BETA= 60.00000	YMIN= 0.00000	YMAX= 400.00000
GAMMA=45.00000	ZMIN= 22.35601	ZMAX= 52.19995

BLOCK 2

Figure 5.5

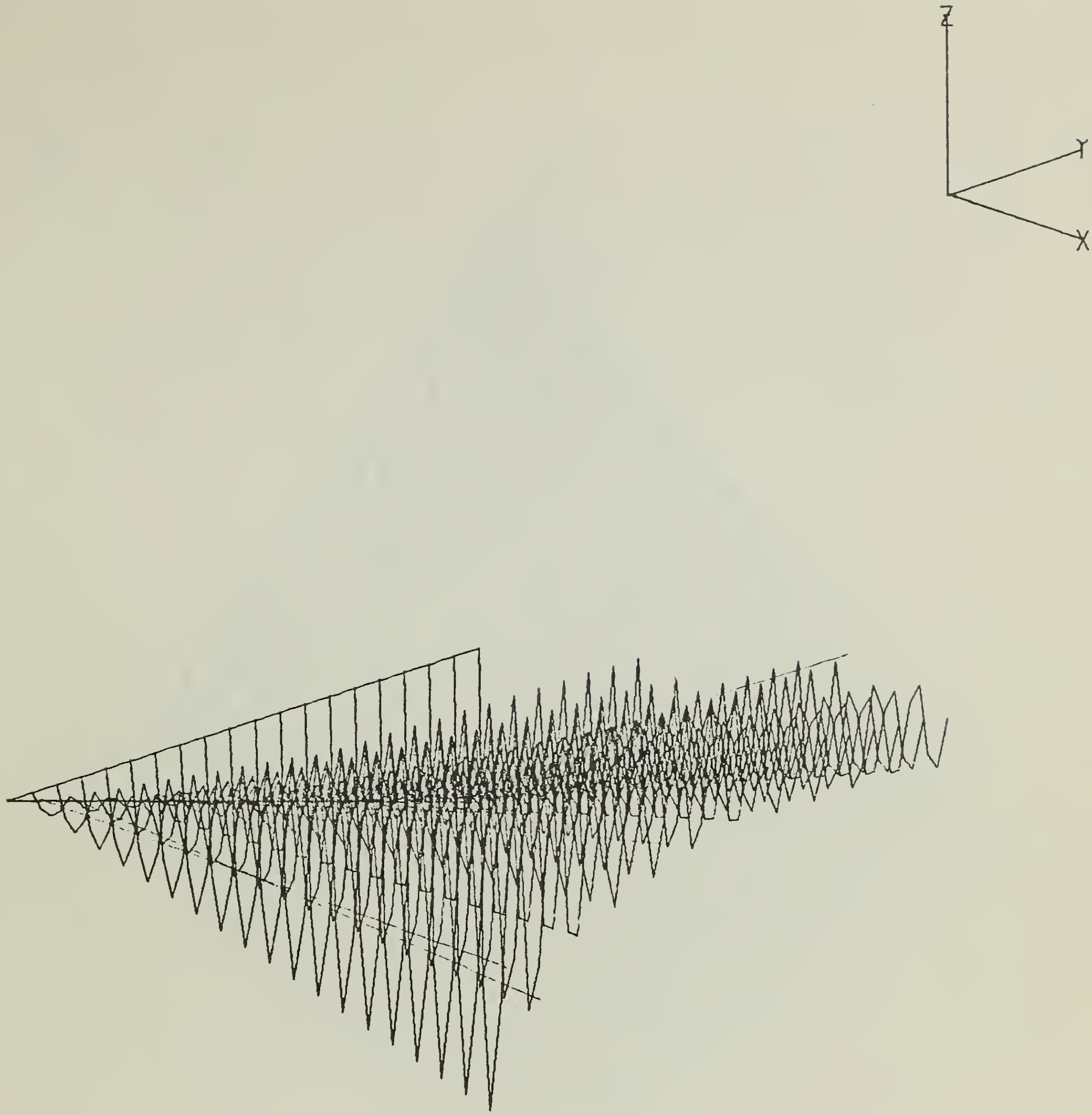


Z = 45. + .06 * X - .056 * ABS(COS(X)) * Y / (1 + .0283 * X)

ALPHA=0.00000	XMIN= 0.00000	XMAX= 120.00000
BETA= 180.00000	YMIN= 0.00000	YMAX= 400.00000
GAMMA=45.00000	ZMIN= 22.35601	ZMAX= 52.19995

BLOCK 5

Figure 5.6



Z = 45. + .06 * X - .056 * ABS(SIN(X)) * Y / (1 + .0283 * X)

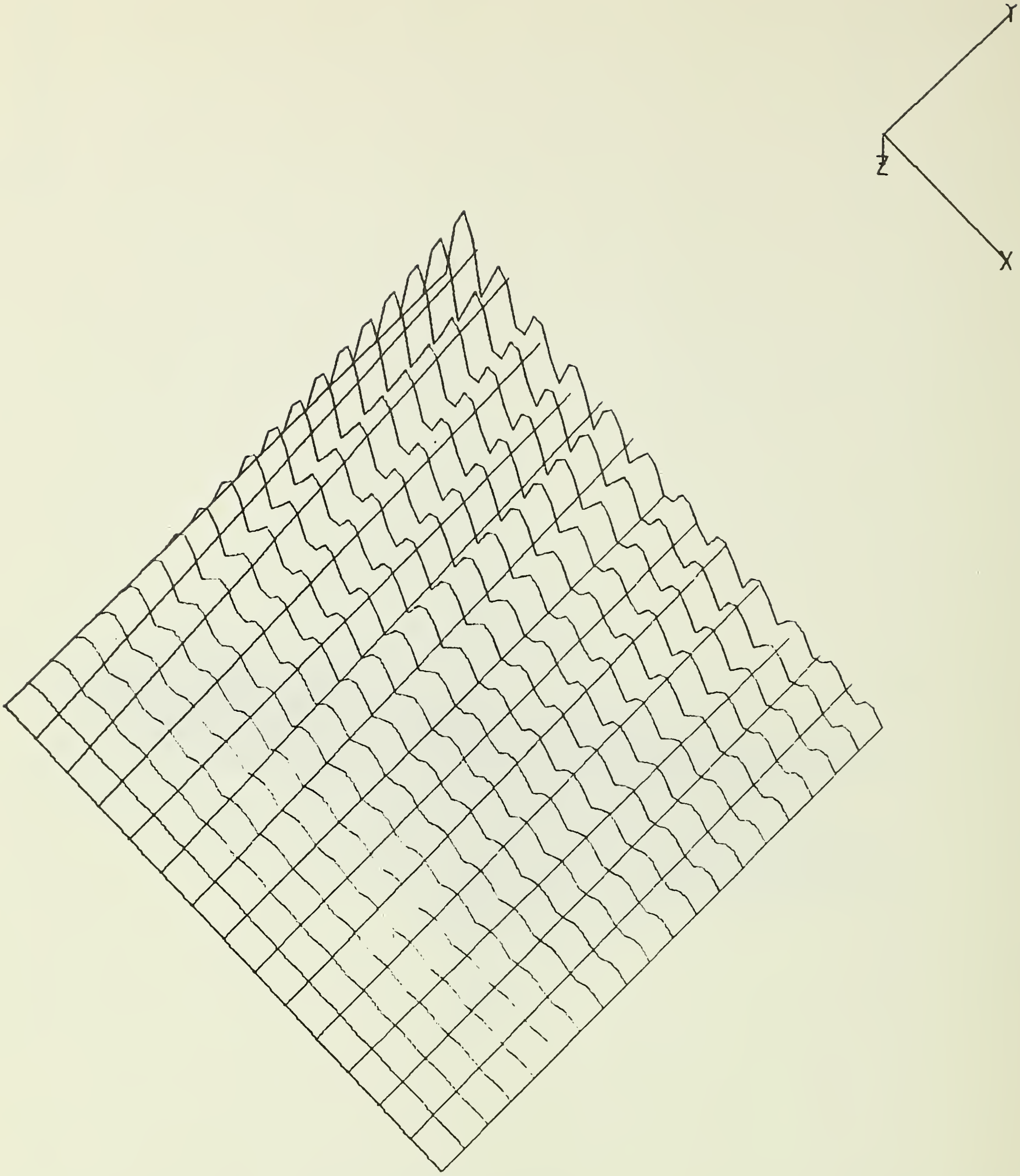
ALPHA=0.00000 XMIN= 0.00000 XMAX= 120.00000

BETA= 20.00000 YMIN= 0.00000 YMAX= 400.00000

GAMMA=45.00000 ZMIN= 25.42896 ZMAX= 52.19995

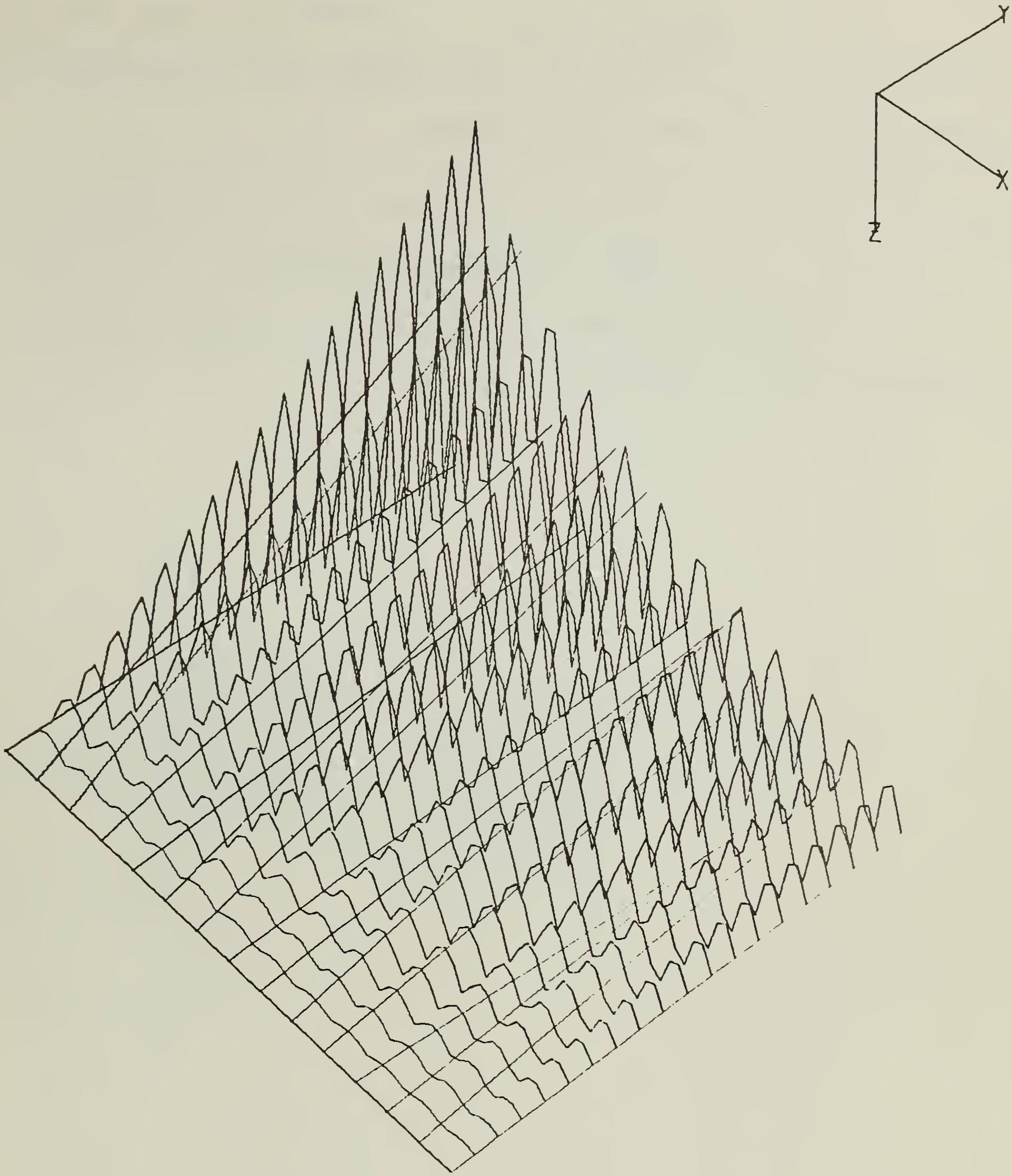
BLOCK 1

Figure 5.7




```
Z = 45. + .06 * X - .056 * ABS(SIN(X)) * Y / (1 + .0283 * X)
ALPHA=0.00000        XMIN= 0.00000        XMAX= 120.00000
BETA= 100.00000      YMIN= 0.00000        YMAX= 400.00000
GAMMA=45.00000       ZMIN= 25.42896       ZMAX= 52.19995
                         BLOCK    3
```

Figure 5.8



Z = 45. + .06 * X - .056 * ABS(SIN(X)) * Y / (1 + .0283 * X)

ALPHA=0.00000	XMIN= 0.00000	XMAX= 120.00000
BETA= 140.00000	YMIN= 0.00000	YMAX= 400.00000
GAMMA=45.00000	ZMIN= 25.42896	ZMAX= 52.19995

BLOCK 4

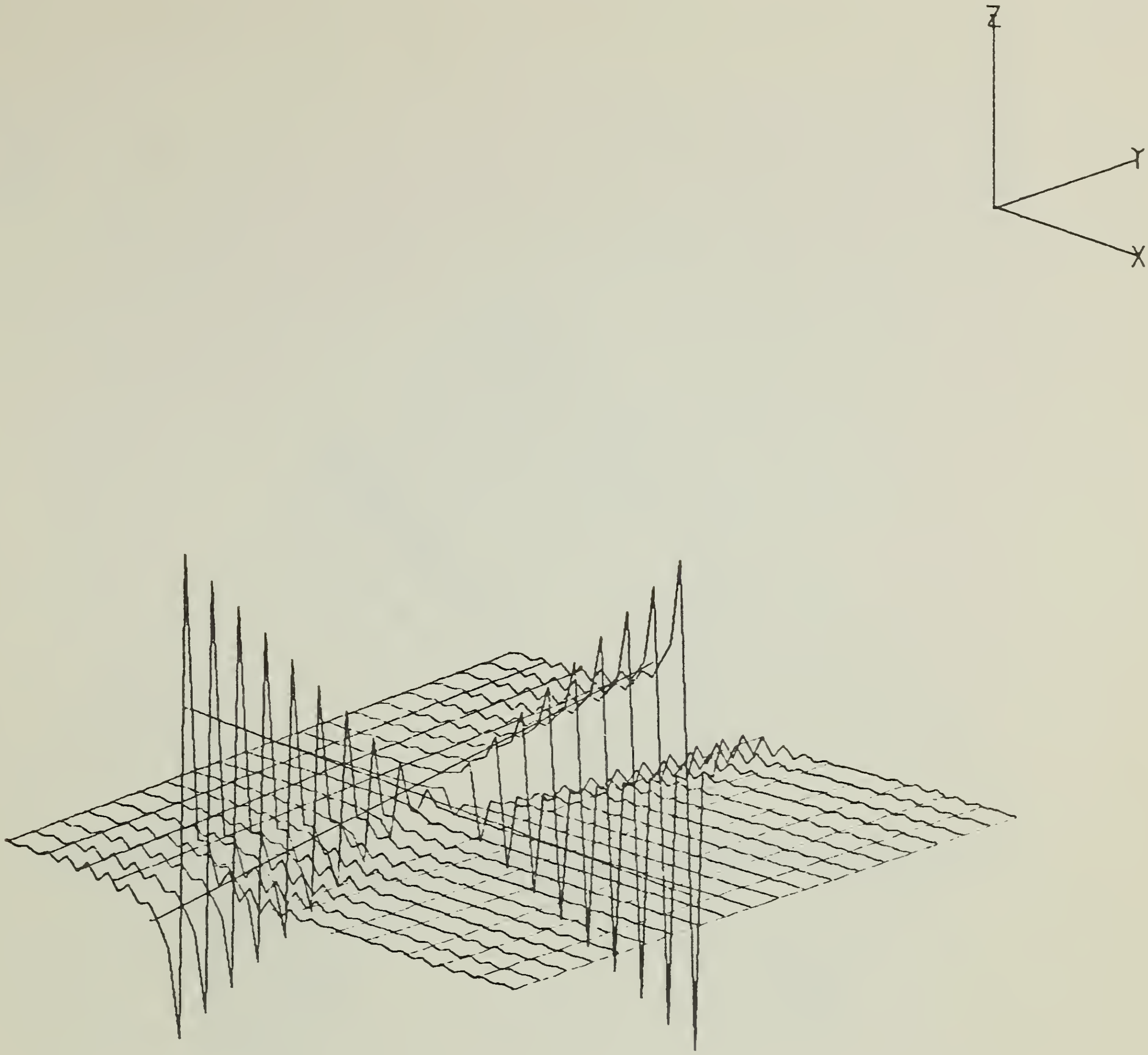
Figure 5.9

(4) Example No. 4. The model is the same as in example No. 3, except that the ranges were extended to the negative directions, as follows,

$$-120 \leq y \leq 120 \text{ ft.}$$

$$-400 \leq q \leq 400 \text{ lb/sq.ft.}$$

Three orientations of the resulting surfaces are given in Figs. 5.10, 5.11, 5.12.



Z = 45. + .06 * X - .056 * ABS(SIN(X)) * Y / (1 + .0283 * X)

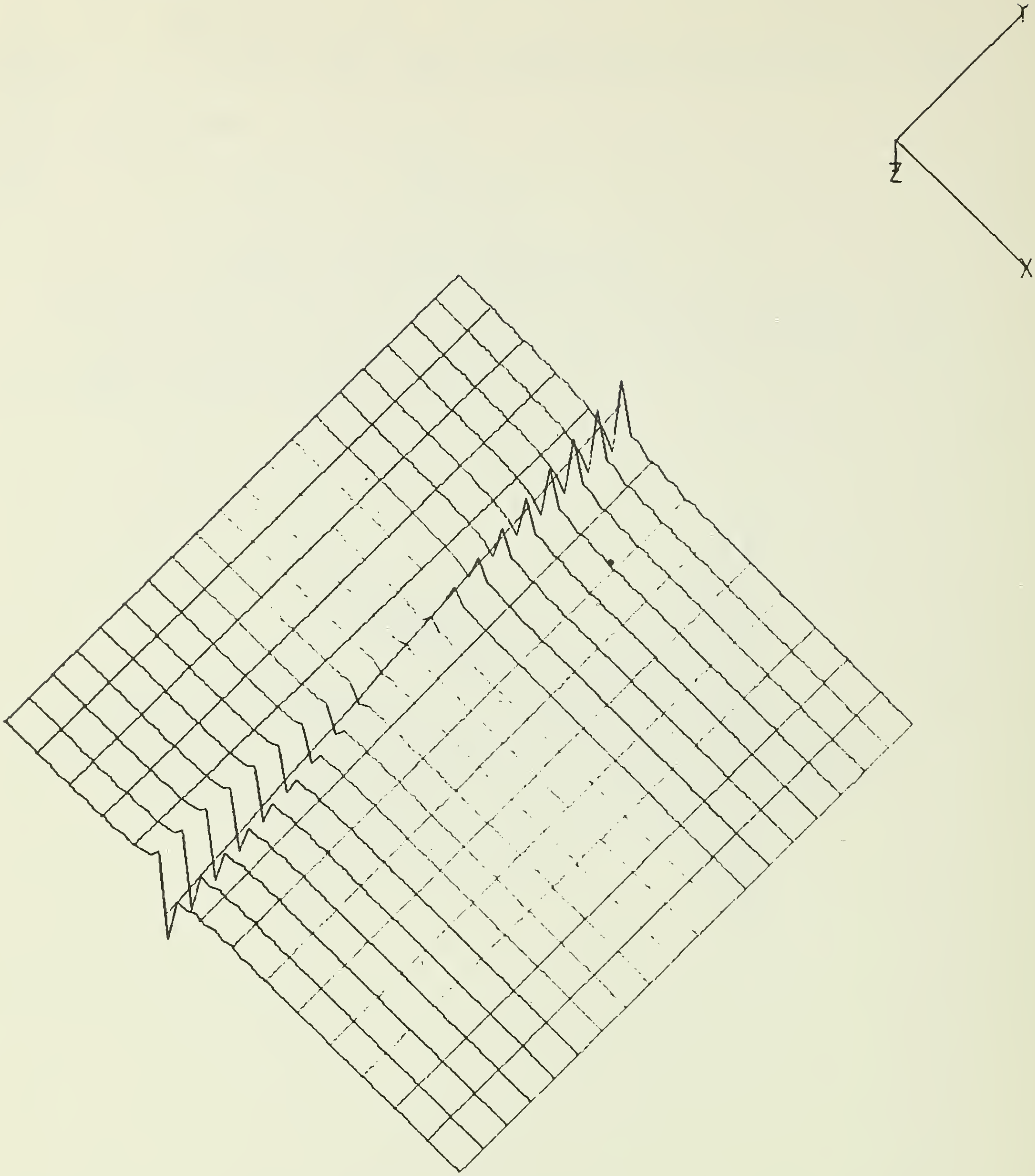
ALPHA=0.00000 XMIN= -120.00000 XMAX= 120.00000

BETA= 20.00000 YMIN= -400.00000 YMAX= 400.00000

GAMMA=45.00000 ZMIN= -335.28848 ZMAX= 421.25616

BLOCK 1

Figure 5.10



Z = 45. + .06 * X - .056 * ABS(SIN(X)) * Y / (1 + .0283 * X)

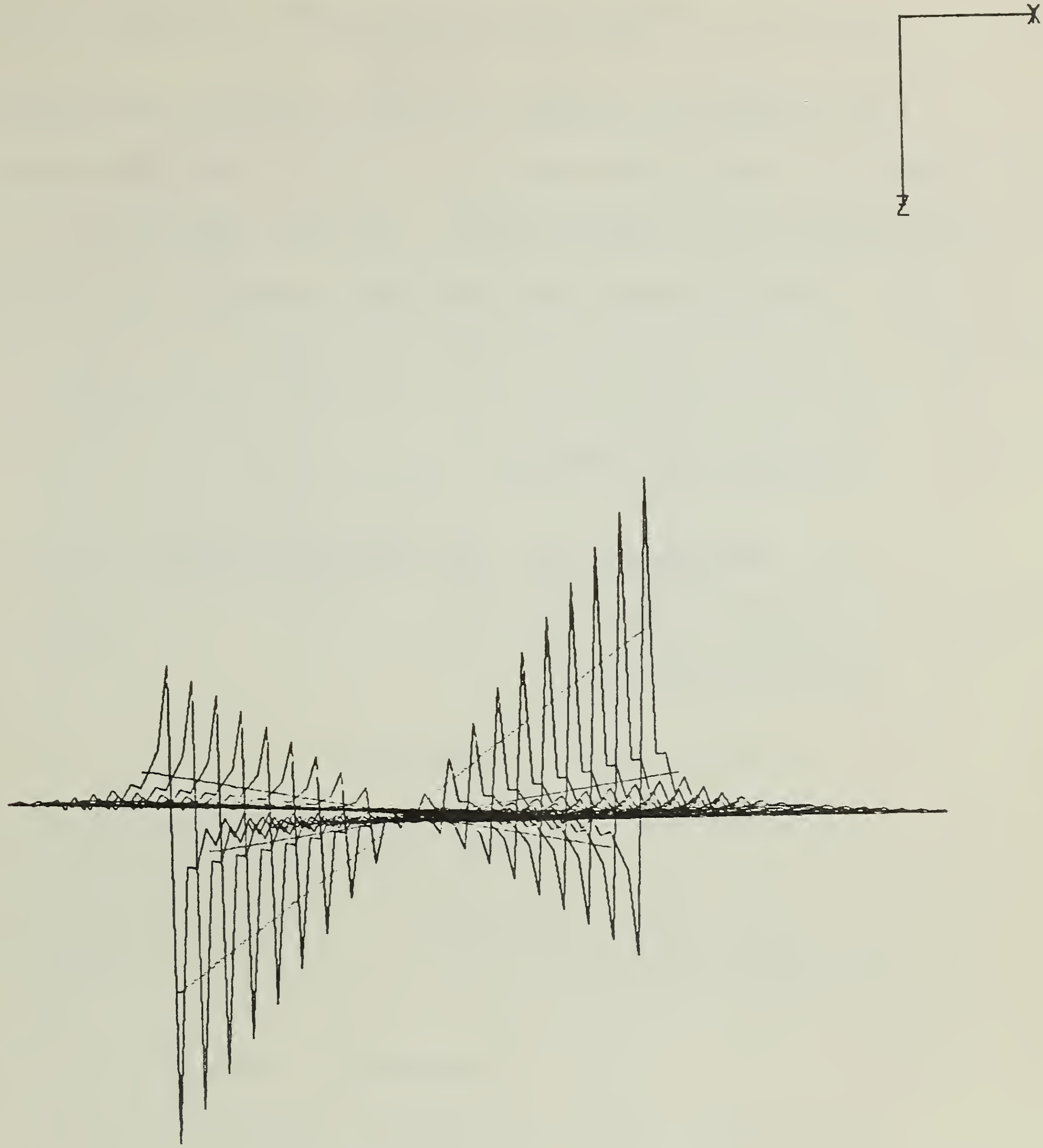
ALPHA=0.00000 XMIN= -120.00000 XMAX= 120.00000

BETA= 100.00000 YMIN= -400.00000 YMAX= 400.00000

GAMMA=45.00000 ZMIN= -335.28848 ZMAX= 421.25616

BLOCK 3

Figure 5.11



Z = 45. + .06 * X - .056 * ABS(SIN(X)) * Y / (1 + .0283 * X)

ALPHA=0.00000 XMIN= -120.00000 XMAX= 120.00000

BETA= 180.00000 YMIN= -400.00000 YMAX= 400.00000

GAMMA=45.00000 ZMIN= -335.28848 ZMAX= 421.25616

BLOCK 5

Figure 5.12

5.3 DIGITAL SIMULATIONS BASED ON THE ALGEBRAIC MODEL

The computer models of this category are based on the algebraic model given by Eq. (3.28). This equation is programmed in subroutine SPH. Various numerical experiments were carried out and some examples and results follow.

5.3.1 Examples and Results

(1) Example No. 1. Janssen-type (static) conditions. The characteristic functions are:

Density-function:

$\gamma = 750$ kg/m^3 (constant).

Ratio-function:

$k = 0.333$ (constant)

Friction-function:

$\lambda = \mu = 0.450$ (constant). Normal friction.

The results give the lateral pressure (kg/m^2) in tabular form for 99 m at 1.0 m intervals.

(2) Example No. 2. Same as example No. 1, but with low friction, i.e., $\mu = 0.225$. The results show that the lateral pressure increase significantly.

DIGITAL SIMULATION

Example No. 1

```
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100)
      READ(1,101) IH,RHO
101   FORMAT(I3,F5.2)
      CALL SDNSTY(DNSTY,IH)
      CALL SRATIO(RATIO,IH)
      CALL SWALL(WALL,IH)
      CALL SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      CALL OUTPT1(PH,IH)
      CALL EXIT
      END
```

```
      SUBROUTINE SDNSTY(DNSTY,IH)
      DIMENSION DNSTY(100)
      DO 25 J=1,IH
25    DNSTY(J)=750.0
      RETURN
      END
```

```
      SUBROUTINE SRATIO(RATIO,IH)
      DIMENSION RATIO(100)
      DO 25 J=1,IH
25    RATIO(J)=0.333
      RETURN
      END
```

```
      SUBROUTINE SWALL(WALL,IH)
      DIMENSION WALL(100)
      DO 25 J=1,IH
25    WALL(J)=0.450
      RETURN
      END
```

```
      SUBROUTINE SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100),XMATRX(100),
     CDET(1100)
      DO 50 K=1,IH
   50 XMATRX(K)=1.0-RATIO(K)*WALL(K)/RHO
      DO 95 IZ=2,IH
      DET(1)=1.0
      IZM1=IZ-1
      DO 75 K=1,IZM1
      I=K+1
      IM1=I-1
      IZMK=IZ-K
   75 DET(I)=DET(IM1)*XMATRX(IZMK)
      SUM=DNSTY(IZ)
      DO 85 K=1,IZM1
      K1=K+1
      IZMK=IZ-K
   85 SUM=SUM+DNSTY(IZMK)*DET(K1)
      PH(IZ)=RATIO(IZ)*SUM
   95 CONTINUE
      RETURN
      END


      SUBROUTINE OUTPI1(PH,IH)
      DIMENSION PH(100)
      WRITE(2,200) (PH(M),M=2,IH)
  200 FORMAT(T30,F10.5)
      RETURN
      END
```

Lateral pressure p(y) (kg./m.^{2})
491.99976
726.97485
954.89380
1175.96802
1390.40332
1598.39819
1800.14819
1995.83887
2185.65283
2369.76660
2548.35229
2721.57471
2889.59521
3052.57007
3210.65015
3363.98364
3512.71216
3656.97412
3796.90430
3932.63159
4064.28369
4191.98047
4315.84375
4435.98828
4552.52344
4665.55859
4775.20312
4881.55078
4984.70703
5084.76562
5181.81641
5275.95703
5367.26562
5455.83594
5541.74609
5625.07422
5705.90234
5784.30078
5860.34766
5934.10937
6005.65625
6075.05469
6142.37109
6207.66016
6270.99609
6332.42578
6392.01172
6449.80469
6505.86719
6560.24219
6612.98828
6664.14453
6713.76953
6761.90234
6808.58984
6853.87500
6897.80078
6940.41016
6981.73437
7021.82031
7060.70703
7098.42187
7135.00000
7170.48437
7204.90234
7238.28906
7270.66797
7302.07812
7332.54297
7362.09375
7390.75781
7418.56250
7445.53125
7471.68750
7497.05859
7521.67187
7545.54297
7568.69531
7591.15625
7612.94141
7634.07031
7654.56641
7674.44531
7693.73047
7712.43359
7730.57422
7748.17187
7765.24219
7781.79687
7797.85937
7813.43359
7828.54297
7843.19922
7857.41406
7871.20312
7884.57812
7897.55078
7910.13281

DIGITAL SIMULATION

Example No. 2

```
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100)
      READ(1,101) IH,RHO
  101 FORMAT(I3,F5.2)
      CALL SDNSTY(DNSTY,IH)
      CALL SRATIO(RATIO,IH)
      CALL SWALL(WALL,IH)
      CALL SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      CALL OUTPT1(PH,IH)
      CALL EXIT
      END
```

```
      SUBROUTINE SDNSTY(DNSTY,IH)
      DIMENSION DNSTY(100)
      DO 25 J=1,IH
   25 DNSTY(J)=750.0
      RETURN
      END
```

```
      SUBROUTINE SRATIO(RATIO,IH)
      DIMENSION RATIO(100)
      DO 25 J=1,IH
   25 RATIO(J)=0.333
      RETURN
      END
```

```
      SUBROUTINE SWALL(WALL,IH)
      DIMENSION WALL(100)
      DO 25 J=1,IH
   25 WALL(J)=0.225
      RETURN
      END
```

```
      SUBROUTINE SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100),XMATRX(100),
     DET(1100)
      DO 50 K=1,IH
   50 XMATRX(K)=1.0-RATIO(K)*WALL(K)/RHO
      DO 95 IZ=2,IH
      DET(1)=1.0
      IZM1=IZ-1
      DO 75 K=1,IZM1
      I=K+1
      IM1=I-1
      IZMK=IZ-K
   75 DET(I)=DET(IM1)*XMATRX(IZMK)
      SUM=DNSTY(IZ)
      DO 85 K=1,IZM1
      K1=K+1
      IZMK=IZ-K
   85 SUM=SUM+DNSTY(IZMK)*DET(K1)
      PH(IZ)=RATIO(IZ)*SUM
   95 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE OUTPT1(PH,IH)
      DIMENSION PH(100)
      WRITE(2,200) (PH(M),M=2,IH)
  200 FORMAT(T30,F10.4)
      RETURN
      END
```

Lateral pressure p(y) (kg./m.^{2})
495.7498
738.0559
976.7239
1211.8083
1443.3621
1671.4397
1896.0925
2117.3723
2335.3284
2550.0132
2761.4734
2969.7598
3174.9177
3376.9954
3576.0381
3772.0933
3965.2046
4155.4141
4342.7695
4527.3125
4709.0820
4888.1250
5064.4766
5238.1836
5409.2812
5577.8086
5743.8086
5907.3125
6068.3633
6226.9961
6383.2461
6537.1523
6688.7461
6838.0625
6985.1367
7130.0039
7272.6953
7413.2422
7551.6836
7688.0430
7822.3555
7954.6484
8084.9609
8213.3125
8339.7383
8464.2656
8586.9219
8707.7383
8826.7422
8943.9570

Lateral pressure p(y) (kg./m.^{2})
9059.4102
9173.1289
9285.1445
9395.4766
9504.1523
9611.1953
9716.6328
9820.4844
9922.7773
10023.5352
10122.7812
10220.5352
10316.8203
10411.6641
10505.0781
10597.0937
10687.7266
10777.0000
10864.9297
10951.5430
11036.8516
11120.8828
11203.6484
11285.1719
11365.4766
11444.5703
11522.4766
11599.2148
11674.8008
11749.2539
11822.5859
11894.8164
11965.9648
12036.0430
12105.0664
12173.0586
12240.0273
12305.9922
12370.9648
12434.9609
12498.0000
12560.0898
12621.2461
12681.4844
12740.8203
12799.2656
12856.8320
12913.5312

(3) <u>Example No. 3</u>. Same as example No. 1, but with alternating friction, i.e.

$$\mu = \begin{cases} 0 \\ 0.450 \end{cases} \quad \text{alternating.}$$

The results are similar to example No. 2. The effect of alternation between 0 and 0.450 is similar to $\mu_{ave.}=0.225$.

(4) <u>Example No. 4</u>. Simulation of the Reimberts' model. The characteristic functions are:

Density-function:

$$\gamma = 750 \quad kg/m^3 \quad \text{(constant)}.$$

Ratio-function:

$$k(y) = \frac{2k_o}{1+(\frac{\mu k_o}{R})y}$$

where,

$k_o = 0.333$

$\mu = 0.225$

$R = 4.99$ m

(5) <u>Example No. 5</u>. Simulation of dynamic pressure curves. Same data as in example No. 4, except for k(y). k(y) as before until y = 75 m., then decreasing at a constant rate of 0.01/m.

DIGITAL SIMULATION

Example No. 3

```
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100)
      READ(1,101) IH,RHO
  101 FORMAT(I3,F5.2)
      CALL SDNSTY(DNSTY,IH)
      CALL SRATIO(RATIO,IH)
      CALL SWALL(WALL,IH)
      CALL SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      CALL OUTPT1(PH,IH)
      CALL EXIT
      END
```

```
      SUBROUTINE SDNSTY(DNSTY,IH)
      DIMENSION DNSTY(100)
      DO 25 J=1,IH
   25 DNSTY(J)=750.0
      RETURN
      END
```

```
      SUBROUTINE SRATIO(RATIO,IH)
      DIMENSION RATIO(100)
      DO 25 J=1,IH
   25 RATIO(J)=0.333
      RETURN
      END
```

```
      SUBROUTINE SWALL(WALL,IH)
      DIMENSION WALL(100)
      DO 25 J=1,IH,2
   25 WALL(J)=0.0
      IHM1=IH-1
      DO 26 I=2,IHM1,2
   26 WALL(I)=0.450
      RETURN
      END
```

```
      SUBROUTINE SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100),XMATRX(100),
     DET(1100)
      DO 50 K=1,IH
   50 XMATRX(K)=1.0-RATIO(K)*WALL(K)/RHO
      DO 95 IZ=2,IH
      DET(1)=1.0
      IZM1=IZ-1
      DO 75 K=1,IZM1
      I=K+1
      IM1=I-1
      IZMK=IZ-K
   75 DET(I)=DET(IM1)*XMATRX(IZMK)
      SUM=DNSTY(IZ)
      DO 85 K=1,IZM1
      K1=K+1
      IZMK=IZ-K
   85 SUM=SUM+DNSTY(IZMK)*DET(K1)
      PH(IZ)=RATIO(IZ)*SUM
   95 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE OUTPT1(PH,IH)
      DIMENSION PH(100)
      WRITE(2,200) (PH(M),M=2,IH)
  200 FORMAT(T30,F10.5)
      RETURN
      END
```

Lateral pressure p(y) (kg./m.^{2})
499.49976
734.24976
983.99951
1204.19995
1453.94922
1660.03589
1909.78442
2102.18433
2351.93286
2531.05444
2780.80298
2947.04419
3196.79272
3350.54395
3600.29248
3741.92554
3991.67407
4121.55078
4371.30078
4489.77734
4739.52734
4846.94922
5096.69922
5193.39453
5443.14453
5529.43750
5779.18359
5865.38672
6105.13281
6171.54687
6421.29297
6478.21094
6727.96094
6775.67187
7025.41797
7064.19531
7313.94141
7344.05469
7593.80469
7615.50781
7865.25781
7878.81250
8128.56250
8134.21094
8383.96094
8381.93750
8631.68750
8622.22656
8871.97266
8855.29687

Lateral pressure p(y) (kg./m.^{2})
9105.04687
9081.37109
9331.11719
9300.65234
9550.40234
9513.35156
9763.10156
9719.66406
9969.41016
9919.77734
*10169.527
*10113.882
*10363.632
*10302.160
*10551.910
*10484.781
*10734.531
*10661.921
*10911.667
*10833.738
*11083.488
*11000.398
*11250.148
*11162.054
*11411.804
*11318.851
*11568.601
*11470.945
*11720.695
*11618.468
*11868.218
*11761.562
*12011.312
*11900.359
*12150.109
*12034.988
*12284.738
*12165.574
*12415.320
*12292.238
*12541.988
*12415.101
*12664.847
*12534.269
*12784.019
*12649.863
*12899.609
*12761.980

DIGITAL SIMULATION

Example No. 4

```
      DIMENSION DNSTY(1CC),RATIO(100),WALL(100),PH(100)
      READ(1,101) IH,RHO
  1C1 FORMAT(I3,F5.2)
      CALL SDNSTY(DNSTY,IH)
      CALL SRATIO(RATIO,IH)
      CALL SWALL(WALL,IH)
      CALL SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      CALL OUTPT1(PH,IH)
      CALL EXIT
      END
```

```
      SUBROUTINE SDNSTY(DNSTY,IH)
      DIMENSION DNSTY(100)
      DO 25 J=1,IH
   25 DNSTY(J)=750.0
      RETURN
      END
```

```
      SUBROUTINE SRATIO(RATIO,IH)
      DIMENSION RATIO(100)
      HYPER=(0.225*C.333)/4.99
      DO 25 J=1,IH
   25 RATIO(J)=0.666/(1.C+HYPER*J)
      WRITE(2,100) (RATIO(J),J=1,IH)
  10C FORMAT(T20,F6.4)
      RETURN
      END
```

```
      SUBROUTINE SWALL(WALL,IH)
      DIMENSION WALL(100)
      DC 25 J=1,IH
   25 WALL(J)=0.225
      RETURN
      END
```

```
      SUBRCUTINE SPH(CNSTY,RATIO,WALL,IH,PH,RHC)
      DIMENSION DNSTY(1CC),RATIO(100),WALL(100),PH(100),XMATRX(100),
     DET() 1100)
      CO 50 K=1,IH
   5C XMATRX(K)=1.0-RATIO(K)*WALL(K)/RHO
      DO 95 IZ=2,IH
      DET(1)=1.0

      IZM1=IZ-1
      CO 75 K=1,IZM1
      I=K+1
      IM1=I-1
      IZMK=IZ-K
   75 DET(I)=DET(IM1)*XMATRX(IZMK)
      SUM=DNSTY(IZ)
      CO 85 K=1,IZM1
      K1=K+1
      IZMK=IZ-K
   85 SUM=SUM+DNSTY(IZMK)*DET(K1)
      PH(IZ)=RATIO(IZ)*SUM
   95 CONTINUE
      RETURN
      ENC




      SUBROUTINE CUTPT1(PH,IH)
      DIMENSION PH(100)
      WRITE(2,200) (PH(M),M=2,IH)
  2CC FORMAT(T30,F10.4)
      RETURN
      END
```

Ratio function k(y)	Lateral pressure p(y) (kg./m.^{2})
0.6561	955.5278
0.6466	1392.3103
0.6373	1804.3472
0.6283	2193.3657
0.6195	2560.9480
0.6110	2908.5542
0.6027	3237.5200
0.5946	3549.0813
0.5867	3844.3767
0.5791	4124.4453
0.5716	4390.2656
0.5643	4642.7187
0.5572	4882.6523
0.5503	5110.8203
0.5436	5327.9375
0.5370	5534.6719
0.5306	5731.6328
0.5243	5919.3828
0.5182	6098.4648
0.5122	6269.3672
0.5063	6432.5586
0.5006	6588.4609
0.4950	6737.4727
0.4896	6879.9961
0.4842	7016.3477
0.4790	7146.8711
0.4739	7271.8867
0.4689	7391.6641
0.4640	7506.4883
0.4592	7616.6055
0.4545	7722.2422
	7823.6445
0.4499	7921.0156
0.4453	8014.5547
0.4409	8104.4453
0.4366	8190.8672
0.4323	8273.9883
0.4281	8353.9531
0.4240	8430.9180
0.4200	8505.0312
0.4161	8576.4023
0.4122	8645.1680
0.4084	8711.4609
0.4047	8775.3672
0.4010	8837.0039
0.3975	8896.4648
0.3939	8953.8594
0.3905	9009.2656
0.3870	9062.7773
0.3837	9114.4609

Ratio function k(y)	Lateral pressure p(y) (kg./m.^{2})
0.3804	9164.4023
0.3772	9212.6719
0.3740	9259.3437
0.3709	9304.4766
0.3678	9348.1445
0.3648	9390.3867
0.3618	9431.2852
0.3589	9470.8789
0.3560	9509.2109
0.3531	9546.3555
0.3504	9582.3437
0.3476	9617.2109
0.3449	9651.0234
0.3422	9683.7891
0.3396	9715.5781
0.3370	9746.4180
0.3345	9776.3320
0.3320	9805.3711
0.3295	9833.5586
0.3271	9860.9336
0.3247	9887.5117
0.3224	9913.3242
0.3200	9938.4062
0.3177	9962.7734
0.3155	9986.4727
0.3132	10009.4961
0.3110	10031.8945
0.3089	10053.6641
0.3067	10074.8594
0.3046	10095.4648
0.3026	10115.5156
0.3005	10135.0352
0.2985	10154.0273
0.2965	10172.5312
0.2945	10190.5547
0.2926	10208.0898
0.2907	10225.1758
0.2888	10241.8320
0.2869	10258.0586
0.2851	10273.8672
0.2832	10289.2891
0.2814	10304.3164
0.2797	10318.9727
0.2779	10333.2656
0.2762	10347.2031
0.2745	10360.7969
0.2728	10374.0664
0.2711	10387.0234
0.2695	
0.2678	

Example No. 5

```
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100)
      READ(1,101) IH,RHO
  101 FORMAT(I3,F5.2)
      CALL SDNSTY(DNSTY,IH)
      CALL SRATIO(RATIO,IH)
      CALL SWALL(WALL,IH)
      CALL SPH(DNSTY,RATIO,WALL,IH,PH,RHC)
      CALL OUTPT1(PH,IH,RATIO)
      CALL EXIT
      END
```

```
      SUBROUTINE SDNSTY(CNSTY,IH)
      DIMENSION DNSTY(100)
      DO 25 J=1,IH
   25 CNSTY(J)=750.0
      RETURN
      END
```

```
      SUBROUTINE SRATIC(RATIO,IH)
      CIMENSION RATIO(100)
      HYPER=(0.225*0.333)/4.99
      DO 25 J=1,75
   25 RATIC(J)=C.666/(1.0+HYPER*J)
      DO 35 J=76,IH
   35 RATIO(J)=RATIC(J-1) -0.C1
      WRITE(2,100) (RATIC(J),J=1,IH)
  100 FORMAT(T20,F6.4)
      RETURN
      END
```

```
      SUBROUTINE SWALL(WALL,IH)
      DIMENSION WALL(100)
      DO 25 J=1,IH
   25 WALL(J)=C.225
      RETURN
      END
```

```
      SUBROUTINE SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100),XMATRX(100),
     DET(1100)
      DO 50 K=1,IH
   50 XMATRX(K)=1.0-RATIO(K)*WALL(K)/RHO

      DO 95 IZ=2,IH
      DET(1)=1.0
      IZM1=IZ-1
      DO 75 K=1,IZM1
      I=K+1
      IM1=I-1
      IZMK=IZ-K
   75 DET(I)=DET(IM1)*XMATRX(IZMK)
      SUM=DNSTY(IZ)
      DO 85 K=1,IZM1
      K1=K+1
      IZMK=IZ-K
   85 SUM=SUM+DNSTY(IZMK)*DET(K1)
      PH(IZ)=RATIO(IZ)*SUM
   95 CONTINUE
      RETURN
      END



      SUBROUTINE CUTPT1(PH,IH,RATIO)
      DIMENSION PH(100),RATIO(100),Z(100)
      WRITE(2,200) (PH(M),M=2,IH)
  200 FORMAT(T30,F10.4)
      Z(1)=1.0
      DO 50 I=1,IH
   50 Z(I+1)=Z(I)+1.0
      CALL MAX(PH(2),98,VALMAX,ISUB)
      CALL MIN(PH(2),98,VALMIN,ISUB)
      CALL PLOT(0,98,2,VALMAX,VALMIN,Z,PH,RATIO)
      RETURN
      END
```

Ratio function k(y)	Lateral pressure p(y) ($kg./m.^2$)	Ratio function k(y)	Lateral pressure p(y) ($kg./m.^2$)
0.6561	955.5278	0.3804	9114.4609
0.6466	1392.3103	0.3772	9164.4023
0.6373	1804.3472	0.3740	9212.6719
0.6283	2193.3657	0.3709	9259.3437
0.6195	2560.9480	0.3678	9304.4766
0.6110	2908.5542	0.3648	9348.1445
0.6027	3237.5200	0.3618	9390.3867
0.5946	3549.0813	0.3589	9431.2852
0.5867	3844.3767	0.3560	9470.8789
0.5791	4124.4453	0.3531	9509.2109
0.5716	4390.2656	0.3504	9546.3555
0.5643	4642.7187	0.3476	9582.3437
0.5572	4882.6523	0.3449	9617.2109
0.5503	5110.8203	0.3422	9651.0234
0.5436	5327.9375	0.3396	9683.7891
0.5370	5534.6719	0.3370	9715.5781
0.5306	5731.6328	0.3345	9746.4180
0.5243	5919.3828	0.3320	9776.3320
0.5182	6098.4648	0.3295	9805.3711
0.5122	6269.3672	0.3271	9833.5586
0.5063	6432.5586	0.3247	9860.9336
0.5006	6588.4609	0.3224	9887.5117
0.4950	6737.4727	0.3200	9913.3242
0.4896	6879.9961	0.3177	9938.4062
0.4842	7016.3477	0.3155	9962.7734
0.4790	7146.8711	0.3132	9735.9375
0.4739	7271.8867	0.3032	9506.0703
0.4689	7391.6641	0.2932	9272.9297
0.4640	7506.4883	0.2832	9036.2266
0.4592	7616.6055	0.2732	8795.7070
0.4545	7722.2422	0.2632	8551.0703
0.4499	7823.6445	0.2532	8302.0547
0.4453	7921.0156	0.2432	8048.3711
0.4409	8014.5547	0.2332	7789.7266
0.4366	8104.4453	0.2232	7525.8242
0.4323	8190.8672	0.2132	7256.3672
0.4281	8273.9883	0.2032	6981.0469
0.4240	8353.9531	0.1932	6699.5391
0.4200	8430.9180	0.1832	6411.5312
0.4161	8505.0312	0.1732	6116.6836
0.4122	8576.4023	0.1632	5814.6602
0.4084	8645.1680	0.1532	5505.0977
0.4047	8711.4609	0.1432	5187.6367
0.4010	8775.3672	0.1332	4861.9141
0.3975	8837.0039	0.1232	4527.5234
0.3939	8896.4648	0.1132	4184.0781
0.3905	8953.8594	0.1032	3831.1609
0.3870	9009.2656	0.0932	3468.3391
0.3837	9062.7773	0.0832	3095.1707
		0.0732	

(6) Example No. 6. Simulation of dynamic pressure curves. Same as example No. 5, but $k(y) = 0.100$ (constant) for $y > 75$ m.

DIGITAL SIMULATION

Example No. 6

```
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100)
      READ(1,101) IH,RHO
  101 FORMAT(I3,F5.2)
      CALL SDNSTY(DNSTY,IH)
      CALL SRATIO(RATIO,IH)
      CALL SWALL(WALL,IH)
      CALL SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      CALL OUTPT1(PH,IH,RATIO)
      CALL EXIT
      END
```

```
      SUBROUTINE SDNSTY(DNSTY,IH)
      DIMENSION DNSTY(100)
      DO 25 J=1,IH
   25 DNSTY(J)=750.0
      RETURN
      END
```

```
      SUBROUTINE SRATIO(RATIO,IH)
      DIMENSION RATIO(100)
      HYPER=(0.225*0.333)/4.99
      DO 25 J=1,75
   25 RATIO(J)=0.666/(1.0+HYPER*J)
      DO 35 J=76,IH
   35 RATIO(J)=0.100
      WRITE(2,100) (RATIO(J),J=1,IH)
  100 FORMAT(T20,F6.4)
      RETURN
      END
```

```
      SUBROUTINE SWALL(WALL,IH)
      DIMENSION WALL(100)
      DO 25 J=1,IH
   25 WALL(J)=0.225
      RETURN
      END
```

```
      SUBROUTINE SPH(DNSTY,RATIO,WALL,IH,PH,RHO)
      DIMENSION DNSTY(100),RATIO(100),WALL(100),PH(100),XMATRX(100),
     DET(1100)
      DO 5C K=1,IH
   50 XMATRX(K)=1.0-RATIO(K)*WALL(K)/RHO


      CO 95 IZ=2,IH
      DET(1)=1.0
      IZM1=IZ-1
      DO 75 K=1,IZM1
      I=K+1
      IM1=I-1
      IZMK=IZ-K
   75 DET(I)=DET(IM1)*XMATRX(IZMK)
      SUM=DNSTY(IZ)
      DO 85 K=1,IZM1
      K1=K+1
      IZMK=IZ-K
   85 SUM=SUM+DNSTY(IZMK)*DET(K1)
      PH(IZ)=RATIO(IZ)*SUM
   95 CONTINUE
      RETURN
      END




      SUBROUTINE OUTPT1(PH,IH,RATIC)
      DIMENSION PH(100),RATIO(100),Z(100)
      WRITE(2,2CC) (PH(M),M=2,IH)
  200 FORMAT(T30,F10.4)
      Z(1)=1.0
      CO 50 I=1,IH
   5C Z(I+1)=Z(I)+1.0
      CALL MAX(PH(2),98,VALMAX,ISUB)
      CALL MIN(PH(2),98,VALMIN,ISUB)
      CALL PLOT(0,98,1,VALMAX,VALMIN,Z,PH)
      RETURN
      END
```

Ratio function k(y)	Lateral pressure p(y) (kg./m.^{2})
0.6561	955.5278
0.6466	1392.3103
0.6373	1804.3472
0.6283	2193.3657
0.6195	2560.9480
0.6110	2908.5542
0.6027	3237.5200
0.5946	3549.0813
0.5867	3844.3767
0.5791	4124.4453
0.5716	4390.2656
0.5643	4642.7187
0.5572	4882.6523
0.5503	5110.8203
0.5436	5327.9375
0.5370	5534.6719
0.5306	5731.6328
0.5243	5919.3828
0.5182	6098.4648
0.5122	6269.3672
0.5063	6432.5586
0.5006	6588.4609
0.4950	6737.4727
0.4896	6879.9961
0.4842	7016.3477
0.4790	7146.8711
0.4739	7271.8867
0.4689	7391.6641
0.4640	7506.4883
0.4592	7616.6055
0.4545	7722.2422
0.4499	7823.6445
0.4453	7921.0156
0.4409	8014.5547
0.4366	8104.4453
0.4323	8190.8672
0.4281	8273.9883
0.4240	8353.9531
0.4200	8430.9180
0.4161	8505.0312
0.4122	8576.4023
0.4084	8645.1680
0.4047	8711.4609
0.4010	8775.3672
0.3975	8837.0039
0.3939	8896.4648
0.3905	8953.8594
0.3870	9009.2656
0.3837	9062.7773
0.3804	9114.4609
0.3772	9164.4023
0.3740	9212.6719
0.3709	9259.3437
0.3678	9304.4766
0.3648	9348.1445
0.3618	9390.3867
0.3589	9431.2852
0.3560	9470.8789
0.3531	9509.2109
0.3504	9546.3555
0.3476	9582.3437
0.3449	9617.2109
0.3422	9651.0234
0.3396	9683.7891
0.3370	9715.5781
0.3345	9746.4180
0.3320	9776.3320
0.3295	9805.3711
0.3271	9833.5586
0.3247	9860.9336
0.3224	9887.5117
0.3200	9913.3242
0.3177	9938.4062
0.3155	9962.7734
0.3132	3210.5771
0.1000	3271.0999
0.1000	3331.3499
0.1000	3391.3284
0.1000	3451.0356
0.1000	3510.4775
0.1000	3569.6450
0.1000	3628.5513
0.1000	3687.1899
0.1000	3745.5623
0.1000	3803.6719
0.1000	3861.5215
0.1000	3919.1113
0.1000	3976.4387
0.1000	4033.5093
0.1000	4090.3198
0.1000	4146.8789
0.1000	4203.1758
0.1000	4259.2266
0.1000	4315.0195
0.1000	4370.5625
0.1000	4425.8555
0.1000	4480.9023
0.1000	4535.6953
0.1000	

CHAPTER 6

MODELS OF HIGHER COMPLEXITIES

6.1 FORMAL GENERALIZATIONS OF THE ONE-DIMENSIONAL MODELS

The one-dimensional models can be generalized formally into higher-dimensional models. The derivations and the results become more complicated and therefore lose some of their practical value. A more practical way of generalization is presented in section 6.2. In the following formal generalizations to two-dimensions are presented.

6.1.1 Analytic Model

In the two-dimensional problem all the variables are considered functions of either the depth coordinate y alone, or both y and the transverse coordinate x.

By a similar development to that presented in section 3.1.2, a solution parallel to Eq. (3.13) is obtained in the form

$$p(\rho,y) = \frac{1}{\rho}\int_{0}^{\rho} \{k(x,y)\,[\exp(-\int\beta(x,y)dy)\,[\int\gamma(x,y)\exp(\int\beta(x,y)dy)dy - -\int\gamma(x,y)\exp(\int\beta(x,y)dy)dy\,|_{y=0}]]\}dx \qquad (6.1)$$

where

$$\beta(x,y) = \frac{\lambda(y)k(x,y)}{\rho}. \qquad (6.2)$$

6.1.2. Algebraic Model

Similarly to the development in section 3.2.1, a net is formed over one-half of the rectangular vertical section of the infinite bin. Starting from the axis of symmetry, v subdivisions, each of width Δx, are made along the x-direction; the vertical partition remains the same as in the one-dimensional case. Each smaller rectangle belonging to the net is double-indexed by i and j where $1 \leq i \leq u$, $1 \leq j \leq v$, and so are the corresponding values of the local parameters. Thus, the parameters are not represented as continuous functions, but as arrays of discrete points which, in turn, can be represented in the form of v-by-u matrices. Then the two-dimensional solution comes out in the form

$$p_{nv} = \rho \sum_{j=1}^{v} K_{nj} \sum_{i=1}^{n} \gamma_{n+1-i,j} \det \mathrm{Sub}_i [I-(\lambda K)_j]$$

or, in terms of k_{nj}

$$p_{nv} = \Delta y \sum_{j=1}^{v} k_{nj} \sum_{i=1}^{n} \gamma_{n+1-i,j} \det \mathrm{Sub}_i [I-\frac{\Delta y}{\Delta x}(\lambda k)_j] \qquad (6.3)$$

where

$$(\lambda k)_j = \begin{bmatrix} 0 & & & & \\ & \lambda_{n-1,j}k_{n-1,j} & & & \mathcal{O} \\ & & \lambda_{n-2,j}k_{n-2,j} & & \\ & & & \ddots & \\ \mathcal{O} & & & & \lambda_{1j}k_{1j} \end{bmatrix}$$

6.1.3 The Characteristic Functions

(1) The density-function. The behavior of the density-function along the y-direction was discussed in section 4.2. The distribution of the density along the x-direction may be assumed to be uniform for most practical cases. However--to be exact--it is probable that the density is larger towards the center line of the bin. If the density distribution in the x-direction is assumed to behave parabolically, and the density distribution in the y-direction is according to Eq. (4.2), then the resulting two-dimensional density-function is illustrated in Figure 6.1.

The derivation of an analytic expression for this function is too complicated and is not necessary. The geometric illustration is qualitative. A more precise diagram can be constructed in any particular case and the entries of the density-matrix can be measured directly from the diagram

and used when evaluating Eq. (4.2).

(2) The ratio-function. The behavior of the ratio-function along the y-direction was discussed in section 4.3. The behavior along the x-direction is even more complicated. There is no experimental data, nor theoretical knowledge of this factor. If assumed constant with respect to the x-direction, a typical two-dimensional ratio-function is illustrated in Figure 6.2; otherwise, it may become as complicated as Figure 6.3 illustrates.

(3) The friction-function. The friction-function remains basically one-dimensional, a case treated in section 4.4.

Figure 6.1.--Two-dimensional density function over a vertical section of the infinite bin.

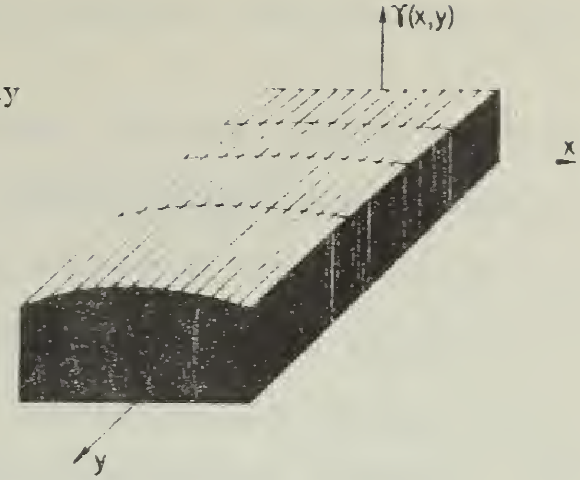


Figure 6.2.--Two-dimensional hyperbolic ratio function over a vertical section of the infinite bin.

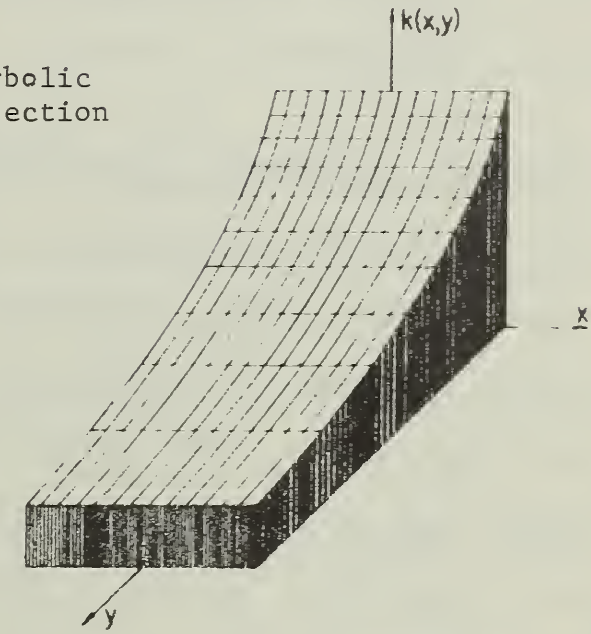
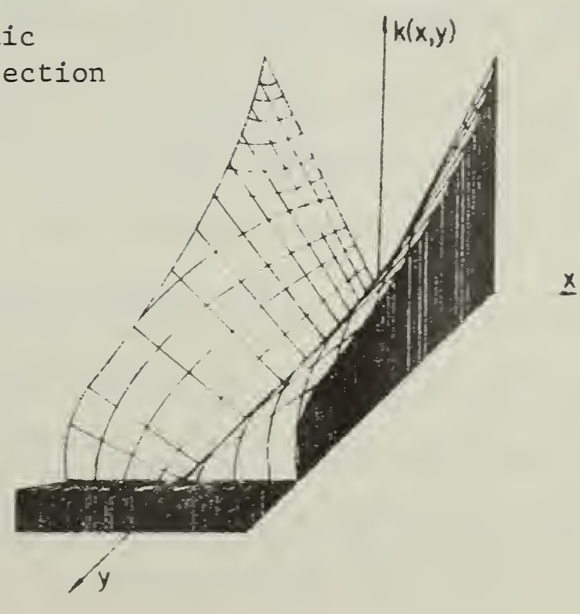


Figure 6.3.--Two-dimensional dynamic ratio function over a vertical section of the infinite bin.



6.2 COMPUTER-ORIENTED GENERALIZATIONS

As seen in the previous section, formal generalizations of the one-dimensional models lead to somewhat complex expressions difficult to handle in practice. An alternate approach is to keep the basic models <u>one-dimensional</u> and reduce higher-dimensional problems to series of one-dimensional problems. It is best to illustrate this technique by a concrete example.

6.2.1 <u>Example: A Square Cross-Section</u>

(1) <u>Janssen's conditions</u>. Consider a bin of a square cross section (Fig. 6.4.a) with sides of length a. Assume static (Janssen's) conditions. All parameters are uniform and constant. For reasons of symmetry treat one quarter only, and within the quarter treat one-half only, Fig. 6.4.b. Subdivide it into a few sections of width $\Delta\omega$ each. The length of each, ρ_i, vary. Each section can be treated as a single section of an infinite bin with $\rho=\rho_i$, and the one-dimensional model can be used repeatedly to calculate the pressure section by section. For example, if a = 16', $\Delta\omega$ = 1', this method requires <u>eight</u> repeated calculations with $\rho_1, \rho_2, \ldots, \rho_8$, compared to just <u>one</u> required by the conventional practice. However, the resolution obtained is quite interesting. Compare the results of the two techniques.

A conventional calculation would give a rectangular pressure diagram, Figure 6.5.a, i.e. uniform lateral pressure across the wall. Examine the results of the new method. The lateral pressure for each section is given by

$$p_i = \frac{\rho_i \gamma}{\mu} [1-\exp\{-(\frac{\mu k}{\rho_i}) y\}].$$

Only ρ_i varies linearly from section to section. It can be seen that p_i is basically directly proportioned to ρ_i because of the dominating $\frac{\rho_i \gamma}{\mu}$ term. This would give a triangular pressure profile. However, this is modified slightly by inverse proportionality due to the appearance of ρ_i in the denominator of the negative power of e. The combined effect is a truncated apex of the triangle, and the profile is closer to a parabola, Fig. 6.5.b.

Now it can be seen that the old method simply <u>averages</u> the triangular profile of the new method. This "averaging" ignores the fact that at the middle the pressure may be nearly twice as high as calculated by the old method. A practical suggestion is to use some compromise, perhaps as in Fig. 6.5.c.

Surcharge can be added in a similar way by segmenting the pyramid over the square cross-section into sections corresponding to the sections beneath. The method can be extended easily to cross-sections of arbitrary shapes.

is uniquely associated with two integers i, j, where $1 \leq i \leq p$, $1 \leq j \leq q$ and pq is the cardinal number of the set A denoted by card A. An example of such a net would be a net of uniform partitions in both directions consisting of intervals of length d, where d is the diameter of the spherical idealized kernels. This net is adopted from now on.

(2) The image set. The associated characteristic half-pile is a right triangle T. It is to be noted that certain regions of the triangle and rectangle overlap. Call the collection of the images of all $a \in A$ (included within the boundaries of T) the image set B. The boundaries of T can be established from two assumptions:

(i) The equality of areas bounded by R and T [4] (i.e., card A = card B).

(ii) The imaginary angle of repose equals the natural angle of repose.

The height h and the base r of T are therefore

$$h = \sqrt{2H\rho \tan\,\theta} \tag{7.1}$$

$$r = \sqrt{2H\rho \cot\,\theta} \tag{7.2}$$

---

[4] An area-preserving mapping, implied by the assumption of a constant density.

(2) Other conditions. If the various parameters are not uniform, the calculation of each section simply utilizes the local parameters independently of neighboring parameters. The total pressure diagram is put together after all sections are calculated. The technique of Appendix B can be used in this connection to give three-dimensional representations of lateral pressure diagrams over the walls.

6.3 FEEDBACK AND COUPLING MECHANISMS

6.3.1 Introduction

Many of the factors entered in the one-dimensional models are interrelated through various direct and indirect relationships. For example, the density, γ, is a function of many factors, including the vertical pressure q, i.e.

$$\gamma = \gamma(q) \ .$$

However, q is or course always a function of γ, i.e.

$$q = q(\gamma) \ .$$

Both are functions of the depth variable y, i.e.

$$\gamma = \gamma(y,q)$$

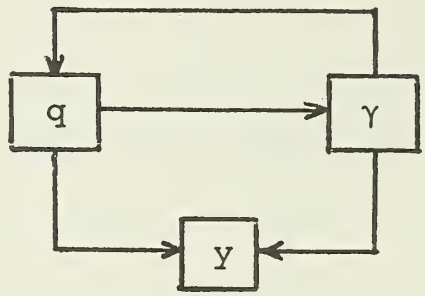
and

$$q = q(y,\gamma(y,q)).$$

Examining the last function, it can be seen that q is ultimately dependent on itself through an indefinite loop that may be written symbolically as,

$$q = q(y,\gamma(y,q(y,\gamma(y,q(....).$$

In diagrammatic form the relationship between q, γ, and y can be presented as follows,



where an arrow pointing from a box to another indicates the dependence of the variable in the first box on the variable in the second. Here, q and γ are dependent variables and y is an indepemdent variable.

The following conventions are adopted:

Dependent variable - at least one arrow originates from box.

Independent variable - no arrow originates from box. (Box only receives arrows).

6.3.2 Modified Incidence Matrix

When many variables are involved, larger and more complex diagrams can be constructed. In a given grain pressure problem, let N be the number of dependent variables, V_1, V_2, ...,

V_N, and M the number of independent variables, $\overline{V}_1$, $\overline{V}_2$, ..., $\overline{V}_M$. Various relationships among variables (both dependent and independent) may hold. For example,

1	$V_i \rightarrow V_j$	for some or all $j \neq i$
0	$V_i \not\rightarrow V_j$	for some or all j
2	$V_i \rightleftarrows V_j$	for some or all $j \neq i$
3	$V_i \rightarrow \overline{V}_k$	for some or all k
0	$V_i \not\rightarrow \overline{V}_k$	for some or all k.

In addition, transitivity holds,i.e., $V_i \rightarrow V_j$ and $V_j \rightarrow V_k$ imply $V_i \rightarrow V_k$.

In order to organize a problem it is best to represent the interrelationships of variables in a modified <u>incidence matrix</u> representing the <u>directed graph</u> associated with the appropriate <u>box diagram</u>. The V_i's are listed horizontally and vertically, followed by the $\overline{V}_i$'s. The numbers 0 through 3 are entered in the intersecting squares according to the given relationships between the variables. This is best illustrated by an example.

Example.

Given: $N = 5$, $M = 2$.

Relationships: $V_1 \rightarrow V_2$, $V_1 \rightleftarrows V_4$, $V_1 \rightleftarrows V_5$, $V_1 \rightarrow \overline{V}_2$, $V_2 \rightarrow V_3$, $V_3 \rightarrow \overline{V}_2$, $V_5 \rightarrow \overline{V}_1$.

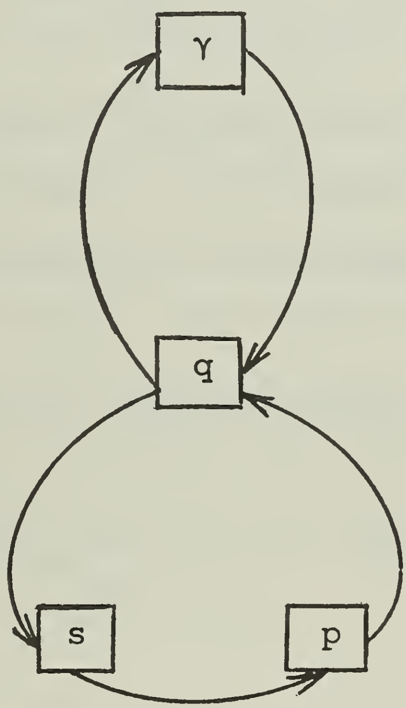
The modified incidence matrix is:

	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	2	2
V_2	1	o	0	0	0
V_3	0	1	0	0	0
V_4	2	0	0	0	0
V_5	2	0	0	0	0
$\overline{V}_1$	0	0	0	0	3
$\overline{V}_2$	3	3	3	3	3

The usefulness of this matrix is in that it codes the interrelationships of the entire system in a simple numeric form that can be easily fed to a digital computer, analyzed further and help in the organization of the calculations of certain coupling effects described in the next section.

6.4 AN EXAMPLE OF A COMPLEX COUPLING MECHANISM

The basic parameters of a grain pressure problem are the density, γ; the vertical and the lateral pressures, q and p; and the frictional stress, s--q, p, and s are interrelated via the ratio and the friction functions, k and λ. The interdependence of these parameters can be represented in a box diagram as follows,



Using this box diagram, all the possible coupling effects due to changes (increase or decrease) in the value of each parameter are enumerated as follows:

$$\gamma^{+} \Longrightarrow q^{+} \qquad \gamma^{-} \Longrightarrow q^{-}$$

$$q^{+} \Longrightarrow \gamma^{+} \qquad q^{-} \Longrightarrow \gamma^{-}$$

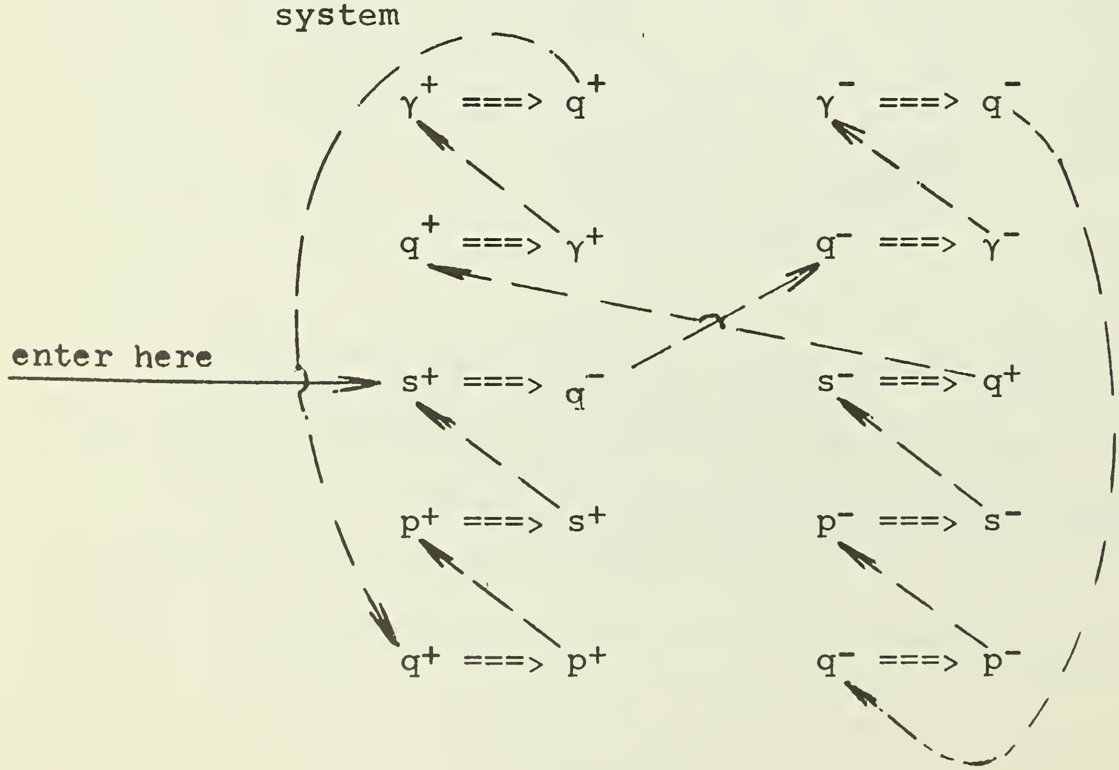
$$s^{+} \Longrightarrow q^{-} \qquad s^{-} \Longrightarrow q^{+}$$

$$p^{+} \Longrightarrow s^{+} \qquad p^{-} \Longrightarrow s^{-}$$

$$q^{+} \Longrightarrow p^{+} \qquad q^{-} \Longrightarrow p^{-}$$

Now, consider the "chain reaction" (fig. 6.6) resulting from a single instantaneous change in λ, a likely occurrence under dynamic conditions. Suppose that λ increases, which causes

Figure 6.6 A path of a "chain reaction" in a grain pressure system



s to increase. Now enter figure 6.6 at s^+ and follow all the resulting changes by drawing connecting arrows. As can be observed, the entire system is interconnected by this chain reaction and finally loops to the starting point, to start a new round. Computationally this means that the entire system has to be recomputed iteratively until relaxation is obtained.

Suppose now that λ decreases, which causes s to decrease. Here the figure should be entered at s^-. If λ^+ and λ^- alternate, as may occur under dynamic conditions, the system oscillates between two entry points s^+ and s^-, while two disparate chain reactions continue to propagate throughout the system.

This analysis shows that a grain pressure system is indeed inherently unstable. Even under static conditions, small changes in λ for example, due to external effects, may cause unpredictable changes throughout the entire system. This may account for some of the observations of small changes in pressures under static conditions.

CHAPTER 7

TOPOLOGICAL MODELS

7.1 THEORY

7.1.1 Introduction: The Associated Characteristic Pile

The theory of grain-pile transformations (GPT) is based on pure mathematical ideas, as well as on the empirical properties of grain under the influence of gravity. It is well known that a mass of grain supports itself in certain unique shapes(such as a cone, a pyramid, etc.) having constant slopes, under natural conditions. The configuration of grain in a deep bin may be considered as a geometrical distortion of the "natural formation" (characteristic pile) which the same mass would have taken, if not constrained by the walls. Thus, a stored mass of grain in a deep bin has a potential tendency to transform into a predictable geometric figure. The pressure exerted on the bin walls may be regarded as a result of this intrinsic tendency.

Therefore, the geometry of a given bin together with the properties of the stored grain are uniquely associated with some characteristic pile. For example, a cylindrical bin is associated with a right circular cone, the dimensions of which are dependent on the dimensions of the circular cylinder, and its slope is dependent on the intrinsic properties of the grain. Factors such as the angle of internal

where

H - height of R

ρ - width of R

θ - natural angle of repose.

Thus T is determined in terms of the bin geometry and the grain properties. Extend the net to include T too, such that

(i) the lower right-hand side vertex is double-indexed by 1,1

(ii) the lower left-hand side vertex is double-indexed by r', 1

(iii) the upper vertex is double-indexed by 1,h'

where

$$r' = \frac{r}{d} \quad \text{and} \quad h' = \frac{h}{d} .$$

(3) Definitions. All terms used in this section are to be understood precisely as defined below, and not as interpreted in general usage.

a. Let $a_{ij} \varepsilon A$. A neighborhood of a_{ij}, designated by $N[a_{ij}]$, is a non-empty subset X of A, such that

(i) $a_{ij} \varepsilon X$

(ii) there exists an $x_{mn} \varepsilon X \ni x_{mn} \neq a_{ij}$

(iii) $x_{mn} \varepsilon X \Longrightarrow x'_{uv} \varepsilon X$, where

$$|i - u| \leq |i - m|$$

$$|j - v| \leq |j - n|.$$

b. A neighbor point of a point $a_{ij} \varepsilon A$ is a point $x_{mn} \neq a_{ij} \varepsilon N[a_{ij}]$ such that

$$|i - m| = 1 \text{ or } 0$$

$$|j - n| = 1 \text{ or } 0$$

c. An interior point of A is a point $a_{ij} \varepsilon A$ that has at least four neighbors.

d. A boundary point of A is a point $a_{ij} \varepsilon A$ that has less than four neighbors.

7.1.3 Grain-Pile Transformations (GPT)

Since card A = card B, there exists a non-empty finite set $\mathfrak{M}$ (card $\mathfrak{M}$ = (pq)!) such that if $M \varepsilon \mathfrak{M}$, then

(i) M is a mapping, $M: A \to B$

(ii) M is one-to-one

(iii) M is onto.

(Note: A mapping M from a set A to a set B is to be one-to-one and onto, if for every element belonging to A there exists a unique element belonging to B, and vice versa.)

Among all $M \varepsilon \mathfrak{M}$, seek a unique mapping M_o such that particular physical constraints are satisfied. The physical constraints are presented in the form of propositions as follows:

PROPOSITION 1: The physical ordering of the initial set A over R is preserved in some unique manner (to be discussed later, see section 7.1.4) in the image set B over T.

PROPOSITION 2: Given any pair $(a_{ij}, M_o(a_{ij}))\varepsilon M_o$, it is possible to find a continuous curve C_{ij} (of non-negative length) joining a_{ij} and its image point $M_o(a_{ij})$, $C_{ij}\varepsilon\mathcal{C}$, card $\mathcal{C}$ = pq, such that if all the a_{ij}'s physically travel along the corresponding C_{ij}'s when a physical transformation actually occurs 5/, some physical factor 6/ is a minimum with respect to any other set $\mathcal{C}'$ (card $\mathcal{C}'$ = pq) of continuous curves C'_{ij} each joining a_{ij} and $M_o(a_{ij})$.

7.1.4 Some Intermediate Results

(1) The decreasing character of the dynamic ratio-function. In line with proposition 1, assume the following properties of the mapping M_o:

5/ Such a physical transformation never does occur in reality during the life-time of the structure. However, when the walls yield to some sufficient extent, the transformation is "nearly" started. The forces acting on each kernel at this instant are tangent to the C_{ij} curves at the initial points a_{ij}. From the point of view of these forces, it is an immaterial coincidence that at the successive instant the walls--rather than proceeding to yield--tend to return. This situation is typical to the dynamic effects during discharge.

6/ In example, work or time. The latter choice suggests a notable resemblance to the classical brachistochrone problem. This one, however, seems to be much more complicated and may be termed therefore as a "finite multi-brachistochrone" problem.

(i) Given $a_{ij} \varepsilon A$, then $\forall\ N[M_o(a_{ij})]$:

(ia) if a_{ij} is a boundary point, $\exists$ a point t in every $N[M_o(a_{ij})] \ni M_o^{-1}(t)$ is a neighbor of a_{ij}.

(ib) if a_{ij} is an interior point, $\exists$ points t_1 and t_2 in every $N[M_o(a_{ij})] \ni M_o^{-1}(t_1)$ and $M_o^{-1}(t_2)$ are distinct neighbors of a_{ij}.

(ii) $\exists$ a non-empty proper subset $I \subset M_o \ni (a_{ij}, M_o(a_{ij})) \varepsilon I \Longrightarrow a_{ij} = M_o(a_{ij})$.

(iii) Boundary conditions for the image set:

(iiia) $(a_{11}, M_o(a_{11})) \varepsilon I$

(iiib) $(a_{1h'}, M_o(a_{1h'})) \varepsilon I$

(iiic) $M_o(a_{pq}) = b_{r'1}$.

(iv) If (i) is not satisfied for some $M_o(a_{ij}) \varepsilon B$, then such an image point is called a point of discontinuity, or a singular point of B.

These properties almost determine the mapping M_o. The second proposition will be used to complete the determination of M_o. In light of proposition 2, $\mathcal{C}$ is a family of mutually non-intersecting catenary-type curves.

Knowing M_o and $\mathcal{C}$, it is possible to find the tangent of the angle (α_{ij}) between the tangent line to C_{ij} at a_{ij} and the vertical direction. The ratio-function at a_{ij} is directly proportional to tan α_{ij}.

The assumed properties of M_o imply the following:

(i) $\mathcal{R}(I)$ 7/ is a non-empty proper subset of B which is in the immediate vicinity of the bottom and the right-hand boundary of R.

(ii) There is a proper subset $S \subset B$ consisting of points of discontinuity of the approximate form as shown in Fig. 7.1, with a "center of singularity" at the middle.

(iii) The remainder of B is a region where the first property of M_o is satisfied.

For illustrative purposes, reduce the two-dimensional problem to a one-dimensional one. Thus the set A is condensed into a set A' of equally-spaced points along the vertical center line R (designated by R'), and the set B is similarly condensed into a set B' along the median of T drawn from the left-hand side vertex of T (designated by T'). The mapping M_o implies that the upper point in A' goes to the far left-hand side of T', the next point in A' goes to the next location on T' and so on, where the spacing between the points along the median is determined by the equality of the areas of the trapezoids associated with the points of B'. Let a' be any point in A' and let k(a') define the (one-

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7/ Range of I.

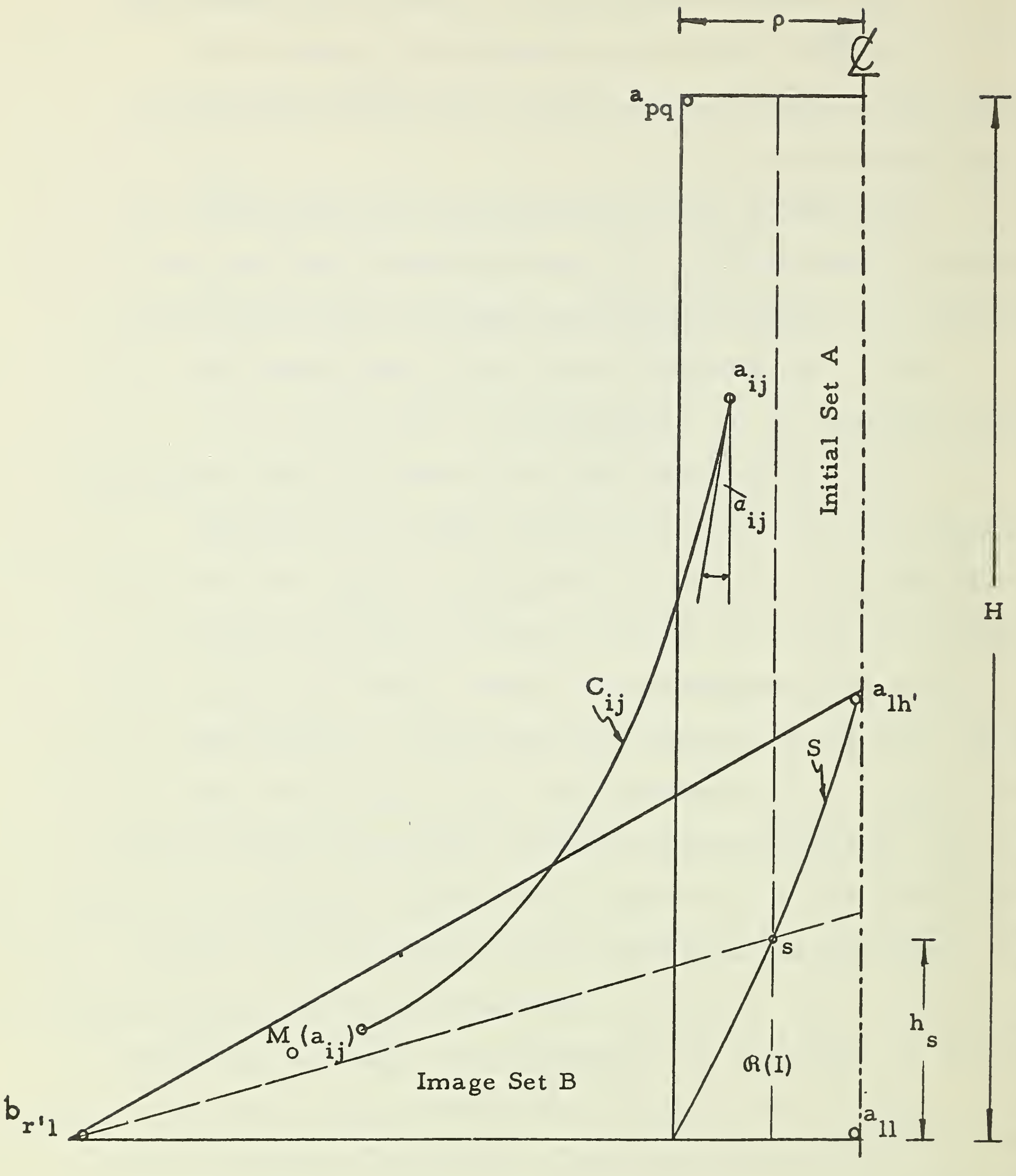


Figure 7.1 Geometrical representation of a two-dimensional grain-pile transformation of the infinite bin.

dimensional) ratio-function at a'. Joining the pairs (a', M_o(a')) by a family of mutually non-interesting catenaries, it is observed that tan α' is monotonically decreasing versus the depth, until the point of intersection (s) between R' and T' is reached. At this point, tan α' is undefined. The decrease of tan α' is emphasized as the point s is approached from above, starting at a height of approximately 2 h_s above the bottom (where h_s is the height of the center of singularity s, above the bottom).

Since k(a') $\propto$ tan α', k(y) has the same qualitative behavior as described for tan α'. This result establishes the decreasing nature of the dynamic k(y) with depth.

(2) <u>The location of the center of singularity</u>. By geometrical considerations

$$h_s = \begin{cases} \sqrt[2]{\frac{1}{2} H\rho \tan\theta} - \frac{\rho}{4}\tan\theta & \text{(a) for the infinite bin} \\ \sqrt[3]{\frac{3}{32} D^2 H \tan^2\theta} - \frac{D}{8}\tan\theta & \text{(b) for a circular cylindrical bin} \end{cases} \qquad (7.3)$$

No comparison between the infinite bin and practical cases can be made; however, a comparison with the circular cylindrical bin is in order.

Consider a typical circular cylindrical grain bin:

$$\frac{H}{D} = 3 \quad \therefore \quad D = \frac{H}{3}$$

$$\theta = 30^{\circ} \quad \therefore \quad \tan\theta = 0.577.$$

Insert these values in Eq. (7.3a) to obtain

$$h_s = 0.127\ H. \tag{7.4}$$

That is, the lateral pressure decreases rapidly toward an elevation which is about 12% above the bottom. The maximum pressure occurs above this elevation along a region whose length is 1½ to 2 h_s, or 0.18H to 0.24H which is altogether 0.30 H to 0.36 H, or <u>one-third</u> of the total height above the bottom.

7.2 TOWARD COMPUTER IMPLEMENTATION OF GPT

Formulating this theory in topological terms it became increasingly more difficult to preserve the applied aspects of the theory. Therefore, a search was started for ways of preserving both the applied aspects and the theoretical concepts of the GPT, while by-passing the abstract notation. The direction to look into was chosen in the field of numerical analysis and computer algorithms. It was found that certain numerical techniques coupled with some unique computer techniques, developed recently at the Los Alamos

Scientific Laboratories, can be useful toward this end. The method can be described roughly as a hybrid between digital simulation techniques and the more traditional finite-difference techniques. It can carry out calculations of non-steady phenomena and output the results as time-dependent realistic motion pictures.

In relation to this problem, the actual mechanism of GPT can be computed and the transformation displayed as a motion picture. By-products of such simulations would be various models of dynamic ratio-functions.

A complete library (51 programs; more than 6000 IBM cards) of a Stromberg-Carlson 4020 Micrifilm Recorder was acquired, edited, compiled, debugged, tested, and made compatible with an IBM 360/50 HASP system. The combined system has the capability to simulate GPT and produce the results as 16 mm motion pictures.

CHAPTER 8

RECOMMENDATIONS FOR FURTHER STUDY

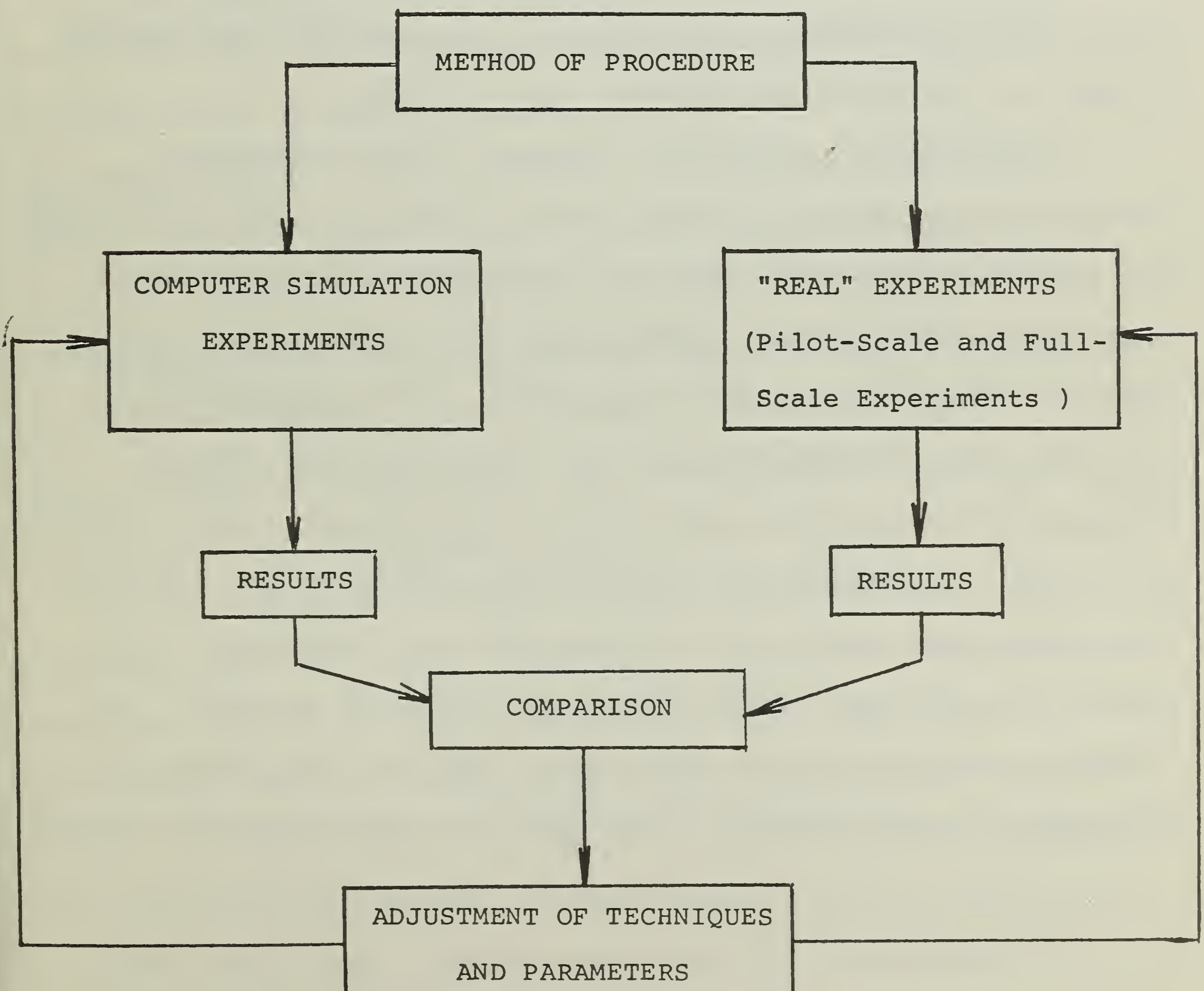
8.1 INTRODUCTION: COMPUTER SIMULATION VERSUS EXPERIMENTATION

Computer simulation has come into increasingly widespread use to study the behavior of complex systems whose state changes over time. Alternatives to the use of simulation are mathematical analysis, experimentation with either the actual system or a prototype of the actual system, or reliance upon experience and intuition. All, including simulation, have limitations. Mathematical analysis of complex systems such as the grain-bin-system is very often impossible; experimentation with actual or pilot systems is costly and time consuming, and the relevant parameters are not always subject to control. Intuition and experience are often the only alternatives available, but can be very inadequate.

Simulation problems are characterized by being mathematically intractable and having resisted solution by analytic methods. The problems usually involve many variables, many parameters, functions which are not well-behaved mathematically, and random variables. The simulation is often a technique of last resort. Yet, much effort is now devoted to computer simulation because it is a technique that gives answers in spite of its difficulties, cost, and time.

During this study various computer simulation techniques were developed and exploited with great success. However,

at various points (for example, bottom of page 44) it was pointed out that ultimately empirical determination of certain parameters must be made and then fed into the basic model in order to establish practical solutions. In fact, it is desirable that both simulation "experiments" and real experiments be carried out, the results compared, the techniques adjusted, etc., until satisfactory solutions are obtained. This procedure is summarized in the following diagram.



As can be seen in the diagram, a dual computer <u>and</u> experimentation approach is proposed for the study of grain pressures. On the one hand various computer simulation experiments are carried out, involving choices of geometries, characteristic functions and other parameters. On the other hand, actual experimentation is carried out, also involving various conditions. The results of the computer analyses and the actual experiments are compared, and the differences between predicted and observed results are noted. These, in turn, indicate changes to be made in the computer experiments as well as the real experiments, and so forth.

The desired goal is, of course, a self-consistent theory encompassing all known factors affecting grain pressures. The computer simulation and the engineering experimentation results are integrated, synthesized, and tested in an overall computer simulation of grain pressures. The simulation has the advantage of demonstrating the total relation of many and varied factors that contribute to the overall grain pressure mechanism; it enables the quantitative study of the relationship among many parts and factors, and the evaluation of the contribution of any single factor to the total picture; in addition, the simulation enables the study of the dynamic or changing situation which is intrinsic to grain pressure systems.

8.2 RECOMMENDATIONS FOR EXPERIMENTAL INVESTIGATIONS

8.2.1 Grain Mechanics

a. General. Conduct laboratory investigations on the geometric, mechanical and rheological properties of grain, by methods analogous to those of soil mechanics, rheology and related fields. Also, investigate the special properties of grain due to organic and biological factors, as affecting grain pressures in storage structures.

b. Specific. The following is a list of some grain properties that were found to be of some significance in studying grain pressure effects:

The single kernel: Average weight, average volume, specific weight, average surface area, typical geometric form, surface roughness, hardness (friction resistance), breakage and mechanical damage, other irregularities, modulus of elasticity, compressive strength, shear strength, impact strength.

Bulk of grain: Unit density, void ratio and porosity, angle of natural slope, internal friction, coefficients of friction on various materials, moisture content (under various temperatures and humidities), modulus of elasticity, compressive strength, static and dynamic shear strength, modulus of resilience, modulus of toughness, stress relaxation, flow-factor, susceptibility to vibration, consolidation, specific heat, diaelectric constant, thermal conductivity, coefficient of volume expansion, organic and biological variations in

volume, other biological properties (such as: absorption of moisture, generation of heat, carbon dioxide, etc.), percentage of foreign matter and dust.

8.2.2 Full-Scale Test Structures

It is recommended to design and construct test bins to carry out series of controlled experiments. A typical test bin should be of circular cross-section, built of reinforced concrete or steel, and have sufficiently large dimensions, i.e., at least 15 ft. diameter and 75 ft. height. Provisions for changing the hopper should be incorporated into the design. At a minimum, the test bin should be instrumented for measuring the following parameters:

(1) lateral pressure

(2) vertical load on the walls

(3) vertical and lateral pressure within the grain

(4) bottom pressure

(5) grain density

The measuring devices should be of adequate range and sensitivity and suitable to record time-dependent variations in pressures. Their number and distribution should be adequate to cover the entire system and to provide good resolution, capable of matching the resolution of the computer simulation.

The results of the measurements should be used to determine empirically the characteristic functions under various conditions, static and dynamic. In particular,

a series of experiments should be conducted to satisfy the following specific objectives:

a. Charging - To determine the relation of:

(1) The point of charging - center or side

(2) Method of charging - continuous or intermittent

(3) Rate of charging - various levels, constant or varying

(4) "Rain charging" - various methods

(5) Inclined flow charging through the wall

(6) Charging by special mechanical devices to reduce kinetic energy. For example: perforated pipe according to the Reimberts' method,

to the vertical loads and lateral and bottom pressures, in the static and dynamic states.

b. Discharging - To determine the relation of:

(1) The point of discharging - central, to the side, at the wall, many gates at various distributions

(2) Various hopper slopes

(3) Method of discharging - continuous or intermittent

(4) Rate of discharging - various levels, constant or varying

(5) Perforated discharging pipe - central or side

(6) Horizontal rings along the perimeter - various dimensions and spacings

(7) Various conditions of gate shut-off time,

to the vertical loads and lateral and bottom pressures, in the static and dynamic states.

c. Flow Pattern - To determine the relation of the flow pattern during charging and discharging to the above mentioned loads and pressures.

d. Simultaneous Charging and Discharging - To determine the relation of simultaneous charging and discharging to the above mentioned loads and pressures.

e. Grains - To determine the relation of:

(1) Various types of grain

(2) Various densities

(3) Coefficient of friction of grain on grain

(4) Moisture content

(5) Cracked grain and foreign material,

to the above mentioned loads and pressures.

f. Aeration - To determine the relation of:

(1) Grain temperature (changes caused during cooling and warming)

(2) Small changes in grain moisture content (both for increasing and decreasing the moisture content),

to the vertical loads and lateral and bottom pressures in the static state.

g. Vibration - To determine the effect of forced vibration producing settling on the vertical loads and lateral and bottom pressures, in the static and dynamic states.

h. Wall Surface - To determine the effect of various wall surfaces:

(1) Smooth steel

(2) Smooth and rough coating on steel

(3) Concrete,

on the vertical loads and lateral and bottom pressures, in the static and dynamic states.

i. Period of Storage - To determine the effect of long storage periods (a year or longer), on the vertical loads and lateral and bottom pressures in the static state.

j. Solar Radiation - To determine the effect of solar heating on the vertical loads and lateral and bottom pressures in the static state.

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APPENDIX A

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APPENDIX B

PROGRAM LISTINGS FOR THREE-DIMENSIONAL PLOTS

Under this project, a computer-based automatic system to plot any two-variable function as a three-dimensional surface was implemented. The system is an adaptation from a NASA Technical Memorandum (TM x-1598, June 1968). It utilizes an IBM 360/50 computer and a Calcomp digital incremental plotter. The software is written in Fortran IV.

In section 5.2 various analytic models are displayed graphically as three-dimensional surfaces. Figures 5.1 through 5.12 are examples of the capabilities of the system. The same system can be used to describe lateral pressure diagrams as three-dimensional surfaces over the walls, two-dimensional characteristic functions, etc.

The program listings follow. In addition, the following standard Calcomp subroutines are assumed to be available: PLT770, NUMBER, SYMBOL.

```
      EXTERNAL FB
      LOGICAL CUBE,DRAWME
      DIMENSION S(6000),IBUFF(6000)
      CALL PLOTS(IBUFF,6000,1)
      CALL PLOT(0.0,-30.0,3)
      CALL PLOT(0.5,-29.5,-3)
      CUBE = .TRUE.
      CALL PLOT3D(0.,120.,0.,400.,S,51,31,7,51,FB,CUBE)
      DRAWME = .FALSE.
      CALL ROTATE(0.0,0.0,45.0,DRAWME)
      DRAWME = .TRUE.
      DO 2 K=1,3
      CALL ROTATE(0.0,20.0,0.0,DRAWME)
      NBL=NBL+1
    2 CONTINUE
      CALL WHERE(XTEM,YTEM,DUM)
      CALL PLOT(XTEM,YTEM,999)
      STOP
      END


      FUNCTION FB(X,Y)
      DATA GAMAO/45./,DELTA/.06/,FKO/.333/,FMU/.425/,R/5./,FLAMDA/.425/
      GAMMA=GAMAO + DELTA*X
      FK = 2.*FKO / (1.+ (FMU*FKO*X/R))
      FB = GAMMA - FLAMDA * FK * Y / R
      RETURN
      END


      BLOCK DATA
      COMMON /XCPIDX/ NBL,RUNNO,CPID(8)
      DATA NBL/1/, RUNNO/4HNO.9/, CPID/4HISTR,7*4H
      END
```

```
      SUBROUTINE PLOT3D(XMIN,XMAX,YMIN,YMAX,S,NXPTS,NYPLS,              PLOT3D 1
     * NXPLS, NYPTS,F,CUBE)                                             PLOT3D 2
      COMMON /LFACT/ FACT
      LOGICAL CUBE                                                      PLOT3D 3
      DIMENSION S(1)                                                    PLOT3D 4
      DATA IK/0/                                                        PLOT3D 5
      IF (IK.NE.0) GO TO 1                                              PLOT3D 6
      CALL WRITE                                                        PLOT3D 7
      IK=1                                                              PLOT3D 8
    1 FNX1 = NXPTS-1                                                    PLOT3D 9
      FNY1 = NYPLS-1                                                    PLOT3D10
      FNX2 = NXPLS-1                                                    PLOT3D11
      FNY2 = NYPTS-1                                                    PLOT3D12
      L1 = NXPTS*NYPLS                                                  PLOT3D13
      L2 = NXPLS*NYPTS                                                  PLOT3D14
      N1=L1                                                             PLOT3D15
      N2=N1+L1                                                          PLOT3D16
      N3=N2+L1                                                          PLOT3D17
      N4=N3+L2                                                          PLOT3D18
      N5=N4+L2                                                          PLOT3D19
      DX=XMAX-XMIN                                                      PLOT3D20
      DY=YMAX-YMIN                                                      PLOT3D21
C                                                                       PLOT3D22
C                                                                       PLOT3D23
      DO 2 I=1,NXPTS                                                    PLOT3D24
      DO 2 J=1,NYPLS                                                    PLOT3D25
      II = NXPTS*(J-1) +I                                               PLOT3D26
      I1=II                                                             PLOT3D27
      I2=N1+II                                                          PLOT3D28
      I3=N2+II                                                          PLOT3D29
      S(I1)=XMIN+DX*FLOAT(I-1)/FNX1                                     PLOT3D30
      S(I2)=YMIN+DY*FLOAT(J-1)/FNY1                                     PLOT3D31
    2 S(I3)=F(S(I1),S(I2))                                              PLOT3D32
C                                                                       PLOT3D33
C                                                                       PLOT3D34
      DO 3 I=1,NXPLS                                                    PLOT3D35
      DO 3 J=1,NYPTS                                                    PLOT3D36
      II = NXPLS*(J-1) + I                                              PLOT3D37
      I1=N3+II                                                          PLOT3D38
      I2=N4+II                                                          PLOT3D39
      I3=N5+II                                                          PLOT3D40
      S(I1)=XMIN+DX*FLOAT(I-1)/FNX2                                     PLOT3D41
      S(I2)=YMIN+DY*FLOAT(J-1)/FNY2                                     PLOT3D42
    3 S(I3)=F(S(I1),S(I2))                                              PLOT3D43
C                                                                       PLOT3D44
C                                                                       PLOT3D45
      N1=N1+1                                                           PLOT3D46
      N2=N2+1                                                           PLOT3D47
```

```
      N3=N3+1                                                           PLOT3D48
      N4=N4+1                                                           PLOT3D49
      N5=N5+1                                                           PLOT3D50
      CALL SCALE(XMIN,XMAX,YMIN,YMAX,S(1),S(N1),S(N2),NXPTS,NYPLS,      PLOT3D51
     * S(N3),S(N4),S(N5),NXPLS,NYPTS,CUBE)                              PLOT3D52
C                                                                       PLOT3D53
C                                                                       PLOT3D54
      CALL AXIS(0,.FALSE.)                                              PLOT3D55
C                                                                       PLOT3D56
C                                                                       PLOT3D57
      ASUM=0.0                                                          PLOT3D58
      BSUM=0.                                                           PLOT3D59
      CSUM=0.                                                           PLOT3D60
      RETURN                                                            PLOT3D61
C                                                                       PLOT3D62
C                                                                       PLOT3D63
      ENTRY ROTATE(ALPHA,BETA,GAMMA,DRAWME)                             PLOT3D64
      LOGICAL DRAWME                                                    PLOT3D65
      IF (FACT .NE. 1.) CALL FACTOR (FACT)
      CALL PLOT(0.0, 0.0, -3)                                           PLOT3D66
      CALL PLOT(0.0, 5.0, -3)                                           PLOT3D67
      IF (FACT .NE. 1.) CALL FACTOR (1.)
      CALL TRNMAT (ALPHA,BETA,GAMMA)                                    PLOT3D68
      CALL PHI  (S(1),S(N1),S(N2),NXPTS,NYPLS)                          PLOT3D69
      CALL PHI  (S(N3),S(N4),S(N5),NXPLS,NYPTS)                         PLOT3D70
      CALL AXIS(1,DRAWME)                                               PLOT3D71
      ASUM=ASUM+ALPHA                                                   PLOT3D72
      BSUM=BSUM+BETA                                                    PLOT3D73
      CSUM=CSUM+GAMMA                                                   PLOT3D74
      IF(.NOT.DRAWME) RETURN                                            PLOT3D75
      CALL DRAWS(S(1),S(N1),S(N2),NXPTS,NYPLS)                          PLOT3D76
      CALL DRAW(S(N3),S(N4),S(N5),NXPLS,NYPTS)                          PLOT3D77
C                                                                       PLOT3D78
C                                                                       PLOT3D79
      CALL WRITES(ASUM,BSUM,CSUM)                                       PLOT3D80
      IF (FACT .NE. 1.) CALL FACTOR (FACT)
      CALL PLOT(10.0,-5.0,-3)                                           PLOT3D81
      IF (FACT .NE. 1.) CALL FACTOR (1.)
      RETURN                                                            PLOT3D82
      END                                                               PLOT3D83
```

```
      BLOCK DATA                                                               1
      COMMON/LABEL/ LAB(18)                                                    2
      COMMON /LISTR/ ISTR
      COMMON /LFACT/ FACT
      DATA ISTR /4HISTR/
      DATA FACT/1./
      DATA LAB /4HISTR, 17*4H
      END                                                                      4
```

```
      SUBROUTINE DRAW (X,Y,Z,NX,NY)                                     DRAW   1
      COMMON /LFACT/ FACT
      IF (FACT .NE. 1.) CALL FACTOR (FACT)
      DIMENSION X(NX,NY), Y(NX,NY), Z(NX,NY)                            DRAW   2
      DO 1 I=1,NX                                                       DRAW   3
      CALL PLOT (Y(I,1),Z(I,1),3)                                       DRAW   4
      DO 1 J=2,NY                                                       DRAW   5
      CALL PLOT (Y(I,J),Z(I,J),2)                                       DRAW   6
1     CONTINUE                                                          DRAW   7
      IF (FACT .NE. 1.) CALL FACTOR (1.)
      RETURN                                                            DRAW   8
      ENTRY DRAWS (X,Y,Z,NX,NY)                                         DRAW   9
      IF (FACT .NE. 1.) CALL FACTOR (FACT)
      DO 2 J=1,NY                                                       DRAW  10
      CALL PLOT (Y(1,J),Z(1,J),3)                                       DRAW  11
      DO 2 I=2,NX                                                       DRAW  12
      CALL PLOT (Y(I,J),Z(I,J),2)                                       DRAW  13
2     CONTINUE                                                          DRAW  14
      IF (FACT .NE. 1.) CALL FACTOR (1.)
      RETURN                                                            DRAW  15
      END                                                               DRAW  16
```

```
      SUBROUTINE WRITE                                                  WRITE  1
      COMMON /LFACT/ FACT
      COMMON /LISTR/ ISTR
      COMMON /MAXES/ XMIN,XMAX,YMIN,YMAX,ZMIN,ZMAX                      WRITE  2
      COMMON /XCPIDX/ NBL,RUNNO,CPID(8)                                 WRITE  3
      COMMON /LABEL/ LAB(18)                                            WRITE  4
      DIMENSION BLOCK(2),ALPHA(2),BETA(2),GAMMA(2),HXMIN(2),HXMAX(2),   WRITE  5
     * HYMIN(2),HYMAX(2),HZMIN(2),HZMAX(2)                              WRITE  6
      DATA BLOCK,ALPHA,BETA,GAMMA/4HBLOC,4HK    ,4HALPH,4HA=  ,4HBETA,4H=WRITE  7
     *   ,4HGAMM,4HA=  /                                                WRITE A7
      DATA HXMIN,HXMAX,HYMIN,HYMAX,HZMIN,HZMAX/4HXMIN,4H=    ,4HXMAX,4H= WRITE B7
     *  ,4HYMIN,4H=    ,4HYMAX,4H=    ,4HZMIN,4H=    ,4HZMAX,4H=        WRITE C7
      IF (CPID(1).EQ.ISTR) RETURN
      IF (FACT .NE. 1.) CALL FACTOR (FACT)
      CALL PLOT (0.0,0.0,-3)                                            WRITE  8
      CALL PLOT (0.0,5.0,-3)                                            WRITE  9
      CPID(8)=RUNNO                                                     WRITE 10
      CALL SYMBOL (0.0,-4.0,.25,CPID,90.0,32)                           WRITE 11
      CALL SYMBOL (0.0,-5.0,0.07,BLOCK,0.0,6)                           WRITE 12
      NBL1=NBL-1                                                        WRITE 13
      FNBL=NBL1                                                         WRITE 14
      CALL NUMBER(0.63,-5.0,0.07,FNBL,0.0,-1)                           WRITE 15
      WRITE (6,2) NBL1                                                  WRITE 16
      CALL PLOT (9.0,-5.0,-3)                                           WRITE 17
      IF (FACT .NE. 1.) CALL FACTOR (1.)
      RETURN                                                            WRITE 18
```

```
      ENTRY WRITES (A1,B1,C1)                                           WRITE 19
      IF (FACT .NE. 1.) CALL FACTOR (FACT)
      A=AMOD(A1,360.0)                                                  WRITE 20
      B=AMOD(B1,360.0)                                                  WRITE 21
      C=AMOD(C1,360.0)                                                  WRITE 22
      CALL SYMBOL (0.0,-5.0,0.07,BLOCK,0.0,6)                           WRITE 23
      FNBL=NBL                                                          WRITE 24
      CALL NUMBER(0.63,-5.0,0.07,FNBL,0.0,-1)                           WRITE 25
      WRITE (6,2) NBL                                                   WRITE 26
      IF (LAB(1).EQ.ISTR) GO TO 1                                       WRITE 27
      CALL SYMBOL (-3.0,-4.0,0.1,LAB,0.0,72)                            WRITE 28
 1    CALL SYMBOL (-3.0,-4.25,0.1,ALPHA,0.0,6)                          WRITE 29
      CALL NUMBER (-2.4,-4.25,0.1,A,0.0,5)                              WRITE 30
      CALL SYMBOL (-3.0,-4.5,0.1,BETA,0.0,6)                            WRITE 31
      CALL NUMBER (-2.4,-4.5 ,0.1,B,0.0,5)                              WRITE 32
      CALL SYMBOL (-3.0,-4.75,0.1,GAMMA,0.0,6)                          WRITE 33
      CALL NUMBER (-2.4,-4.75,0.1,C,0.0,5)                              WRITE 34
      CALL SYMBOL (-1.0,-4.25,0.1,HXMIN,0.0,5)                          WRITE 35
      CALL NUMBER (-0.4,-4.25,0.1,XMIN,0.0,5)                           WRITE 36
      CALL SYMBOL (-1.0,-4.5 ,0.1,HYMIN,0.0,5)                          WRITE 37
      CALL NUMBER (-0.4,-4.5 ,0.1,YMIN,0.0,5)                           WRITE 38
      CALL SYMBOL (-1.0,-4.75,0.1,HZMIN,0.0,5)                          WRITE 39
      CALL NUMBER (-0.4,-4.75,0.1,ZMIN,0.0,5)                           WRITE 40
      CALL SYMBOL ( 1.0,-4.25,0.1,HXMAX,0.0,5)                          WRITE 41
      CALL NUMBER ( 1.6,-4.25,0.1,XMAX,0.0,5)                           WRITE 42
      CALL SYMBOL ( 1.0,-4.5 ,0.1,HYMAX,0.0,5)                          WRITE 43
      CALL NUMBER ( 1.6,-4.5 ,0.1,YMAX,0.0,5)                           WRITE 44
      CALL SYMBOL ( 1.0,-4.75,0.1,HZMAX,0.0,5)                          WRITE 45
      CALL NUMBER ( 1.6,-4.75,0.1,ZMAX,0.0,5)                           WRITE 46
      IF (FACT .NE. 1.) CALL FACTOR (1.)
      RETURN                                                            WRITE 47
C                                                                       WRITE 48
    2 FORMAT (120X,5HBLOCK,I5)                                          WRITE 49
      END                                                               WRITE 50
```

```
      SUBROUTINE SCALE (XMIN,XMAX,YMIN,YMAX,X1,Y1,Z1,NX1,NY1,X2,Y2,Z2,NX
     *2,NY2,CUBE)
      DIMENSION X1(NX1,NY1), Y1(NX1,NY1), Z1(NX1,NY1), X2(NX2,NY2), Y2(N
     *X2,NY2), Z2(NX2,NY2)
      COMMON /MAXES/ XMIN1,XMAX1,YMIN1,YMAX1,ZMIN,ZMAX
      LOGICAL CUBE
      REAL MAXX,MAXY,MAXZ
      XMIN1=XMIN
      XMAX1=XMAX
      YMIN1=YMIN
      YMAX1=YMAX
      MAXX=(XMAX-XMIN)/3.5
      MAXY=(YMAX-YMIN)/3.5
      ZMAX=Z1(1,1)
      ZMIN=Z1(1,1)
      DO 1 I=1,NX1
      DO 1 J=1,NY1
       ZMAX=AMAX1(Z1(I,J),ZMAX)
    1 ZMIN=AMIN1(Z1(I,J),ZMIN)
      DO 2 I=1,NX2
      DO 2 J=1,NY2
      ZMAX=AMAX1(Z2(I,J),ZMAX)
    2 ZMIN=AMIN1(Z2(I,J),ZMIN)
      MAXZ=(ZMAX-ZMIN)/3.5
      IF (CUBE) GO TO 3
      MAXX=AMAX1(MAXX,MAXY,MAXZ)
      MAXY=MAXX
      MAXZ=MAXX
    3 DO 4 I=1,NX1
      DO 4 J=1,NY1
      X1(I,J)=(X1(I,J)-(XMAX+XMIN)/2.)/MAXX
      Y1(I,J)=(Y1(I,J)-(YMAX+YMIN)/2.)/MAXY
    4 Z1(I,J)=(Z1(I,J)-(ZMAX+ZMIN)/2.)/MAXZ
      DO 5 I=1,NX2
      DO 5 J=1,NY2
      X2(I,J)=(X2(I,J)-(XMAX+XMIN)/2.)/MAXX
      Y2(I,J)=(Y2(I,J)-(YMAX+YMIN)/2.)/MAXY
    5 Z2(I,J)=(Z2(I,J)-(ZMAX+ZMIN)/2.)/MAXZ
      RETURN
      END
```

```
      SUBROUTINE AXIS(I,DRAWME)                                         AXIS   1
      COMMON /LFACT/ FACT
      LOGICAL DRAWME                                                    AXIS   2
      DIMENSION X(4,1),Y(4,1),Z(4,1)                                    AXIS   3
      IF (I.NE.0) GO TO 1                                               AXIS   4
      X(1,1)=0.                                                         AXIS   5
      X(2,1)=1.0                                                        AXIS   6
      X(3,1)=0.                                                         AXIS   7
      X(4,1)=0.                                                         AXIS   8
      Y(1,1)=0.                                                         AXIS   9
      Y(2,1)=0.                                                         AXIS  10
      Y(3,1)=1.0                                                        AXIS  11
      Y(4,1)=0.                                                         AXIS  12
      Z(1,1)=0.                                                         AXIS  13
      Z(2,1)=0.                                                         AXIS  14
      Z(3,1)=0.                                                         AXIS  15
      Z(4,1)=1.0                                                        AXIS  16
      RETURN                                                            AXIS  17
    1 CALL PHI(X,Y,Z,4,1)                                               AXIS  18
      IF (DRAWME) GO TO 2                                               AXIS  19
      RETURN                                                            AXIS  20
    2 YY=Y(1,1)+4.0                                                     AXIS  21
      ZZ=Z(1,1)+4.0                                                     AXIS  22
      IF (FACT .NE. 1.) CALL FACTOR (FACT)
      CALL PLOT(YY,ZZ,3)                                                AXIS  23
      CALL PLOT(Y(2,1)+4., Z(2,1)+4., 2)                                AXIS  24
      CALL SYMBOL(Y(2,1)+3.97143, Z(2,1)+3.95, .1, 1HX, 0.0, 1)         AXIS  25
      CALL PLOT(YY,ZZ,3)                                                AXIS  26
      CALL PLOT(Y(3,1)+4., Z(3,1)+4., 2)                                AXIS  27
      CALL SYMBOL(Y(3,1)+3.97143, Z(3,1)+3.95, .1, 1HY, 0.0, 1)         AXIS  28
      CALL PLOT(YY,ZZ,3)                                                AXIS  29
      CALL PLOT(Y(4,1)+4., Z(4,1)+4., 2)                                AXIS  30
      CALL SYMBOL(Y(4,1)+3.97143, Z(4,1)+3.95, .1, 1HZ, 0.0, 1)         AXIS  31
      IF (FACT .NE. 1.) CALL FACTOR (1.)
      RETURN                                                            AXIS  32
      END                                                               AXIS  33
```

```
      SUBROUTINE TRNMAT (ALPHA,BETA,GAMMA)
      COMMON /MATRIX/ TMAT(3,3)
      A=ALPHA/57.2957795
      B= BETA/57.2957795
      C=GAMMA/57.2957795
      SINA=SIN(A)
      SINB=SIN(B)
      SINC=SIN(C)
      COSA=COS(A)
      COSB=COS(B)
      COSC=COS(C)
      TMAT(1,1)=COSC*COSB
      TMAT(1,2)=COSC*SINB*SINA-SINC*COSA
      TMAT(1,3)=COSC*SINB*COSA+SINC*SINA
      TMAT(2,1)=SINC*COSB
      TMAT(2,2)=SINC*SINB*SINA+COSC*COSA
      TMAT(2,3)=SINC*SINB*COSA-COSC*SINA
      TMAT(3,1)=-SINB
      TMAT(3,2)=COSB*SINA
      TMAT(3,3)=COSB*COSA
      RETURN
      END
```

```
      SUBROUTINE PHI (X,Y,Z,NX,NY)
      DIMENSION X(NX,NY), Y(NX,NY), Z(NX,NY)
      COMMON /MATRIX/ TMAT(3,3)
C
C
      DO 1 I=1,NX
      DO 1 J=1,NY
C
C
      XP=TMAT(1,1)*X(I,J)+TMAT(1,2)*Y(I,J)+TMAT(1,3)*Z(I,J)
      YP=TMAT(2,1)*X(I,J)+TMAT(2,2)*Y(I,J)+TMAT(2,3)*Z(I,J)
      ZP=TMAT(3,1)*X(I,J)+TMAT(3,2)*Y(I,J)+TMAT(3,3)*Z(I,J)
C
C
      X(I,J)=XP
      Y(I,J)=YP
    1 Z(I,J)=ZP
C
C
      RETURN
      END
```